I N D E X

TIFR - PHYSICS

CHAPTER-1 MATHEMATICAL METHODS IN PHYSICS

PREVIOUS YEAR EXAM QUESTIONS

VECTOR ALGEBRA & VECTOR CALCULUS

[TIFR 2012]

1. The function $f(x)$ represents the nearest integer less than x, e.g.

$$f(3.14) = 3$$

The derivative of this function (for arbitrary x) will be given in terms of the integers n as

$f'(x) =$

(a) 0 (b) $\sum_n \delta(x-n)$

(c) $\sum_n |x-n|$ (d) $\sum_n f(x-n)$

[TIFR-2013]

2. The value of the integral $\int_0^\infty dx\, x^9 \exp(-x^2)$ is

(a) 20160 (b) 12

(c) 18 (d) 24

[TIFR-2013]

3. The integral $\int_{-\infty}^{\infty} dx \delta(x^2-\pi^2) cos x$ evaluates to

(a) -1 (b) 0

(c) $1/\pi$ (d) $-1/\pi$

[TIFR-2014]

4. In spherical polar coordinates $\vec{r} = (r, \theta, \varphi)$ the delta function $\delta(\vec{r}_1 - \vec{r}_2)$ can be written as

(a) $\delta(r_1-r_2)\delta(\theta_1-\theta_2)\delta(\varphi_1-\varphi_2)$

(b) $\frac{1}{r_1^2}\ \delta(r_1-r_2)\delta(cos\theta_1 - cos\theta_2)\delta(\varphi_1-\varphi_2)$

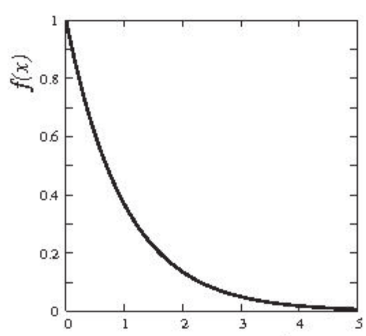
(c) $\frac{1}{|\vec{r}_1-\vec{r}_2|^2}\ \delta(r_1-r_2)\delta(cos\theta_1 - cos\theta_2)\delta(\varphi_1-\varphi_2)$

(d) $\frac{1}{r_1^2 cos\theta_1}\delta(r_1-r_2)\delta(cos\theta_1 - cos\theta_2)\delta(\varphi_1-\varphi_2)$

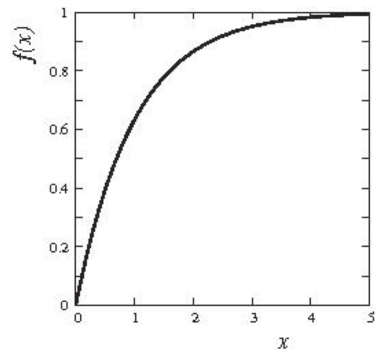
[TIFR-2014]

5. The solution of the integral equation $f(x) = x - \int_0^x dt\, f(t)$has the graphical form

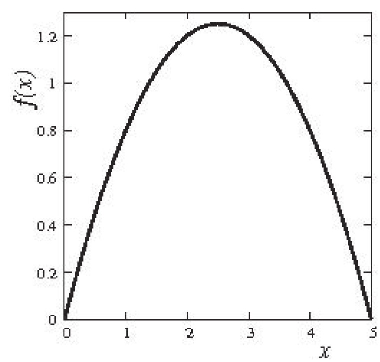
(a)



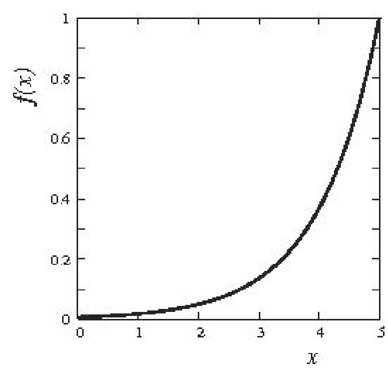
(b)



(c)



(d)



[TIFR-2015]

6. Which of the following vectors is parallel to the surface $x^2y + 2xz = 4$ at the point$(2,-2,3)$?

(a) $-6\hat{\imath} - 2\hat{\jmath} + 5\hat{k}$ (b) $6\hat{\imath} + 2\hat{\jmath} + 5\hat{k}$

(c) $6\hat{\imath} - 2\hat{\jmath} + 5\hat{k}$ (d) $6\hat{\imath} - 2\hat{\jmath} - 5\hat{k}$

[TIFR-2016]

7. If x is a continuous variable which is uniformly distributed over the real line from $x = 0$ to $x \to \infty$ according to the distribution $f(x) = ex(-4x)$ then the expectation value of $\cos 4x$ is

(a) Zero (b) 1/2
(c) 1/4 (d) 1/16

[TIFR-2018]

8. Evaluate the integral
$\int_{-\infty}^{+\infty} dx \exp(-x^2) \cos(\sqrt{2}x)$

[TIFR-2018]

9. A fourth rank Cartesian tensor T_{ijkt} satisfies the following identities
(i) $T_{ijk\ell} = T_{ijk\ell}$
(ii) $T_{ijk\ell} = T_{ij\ell k}$
(iii) $T_{ijk\ell} = T_{k\ell ij}$
Assuming a space of three dimensions ($i.e. i, j, k = 1, 2, 3$), what is the number of independent components of $T_{ijk\ell}$?

[TIFR-2019]

10. Consider the surface defined by $ax^2 + by^2 + cz + d = 0$, where a, b, c and d are constants. If $\hat{n}_1$ and $\hat{n}_2$ are unit normal vectors to the surface at the points $(x, y, z) = (1, 1, 0)$ and $(0, 0, 1)$ respectively and $\hat{m}$ is a unit vector normal to both $\hat{n}_1$ and $\hat{n}_2$, then $\hat{m}$
(a) $\frac{-ai+bj}{\sqrt{a^2+b^2}}$ (b) $\frac{a\hat{\imath}+b\hat{\jmath}+c\hat{k}}{\sqrt{a^2+b^2+c^2}}$
(c) $\frac{b\hat{\imath}-a\hat{\jmath}}{\sqrt{a^2+b^2}}$ (d) $\frac{2a\hat{\imath}+2b\hat{\jmath}-c\hat{k}}{\sqrt{4a^2+4b^2+c^2}}$

[TIFR-2019]

11. An array T has elements T_{ijkl} where $i, j, k, l = 1,2,3,4$. It is given that
$$T_{ijkl} = T_{ijkl} = T_{ijkl} = -T_{klij}$$
for all values of i, j, k, l. The number of independent components in this array is
(a) 55 (b) 256
(c) 45 (d) 1

[TIFR-2020]

12. The limit $\lim_{x\to\infty} x \log\frac{x+1}{x-1}$ evaluates to
(a) 2 (b) 0
(c) ∞ (d) 1

[TIFR-2022]

13. Consider the two-dimensional polar integral
$$P = \int dr d\theta r^{19} e^{-r^2} \sin^8\theta \cos^{11}\theta$$
If the integration is over only the first quadrant ($0 \leq \theta \leq \pi/2$), the value of P is
(a) 20160 (b) 88π
(c) 180 (d) 16π

[TIFR-2022]

14. The value of the integral
$$\int_{-\pi/2}^{+\pi/2} dx \cosh(kx^2) \ \sin^2 x$$
In the large-k limit, will be
(a) $\frac{1}{2k\pi} e^{k\pi^2/4}$ (b) $\frac{1}{k^2\pi^2} cosh\left(\frac{\pi^2}{4}\right)$
(c) $cosh\left(\frac{\pi^2}{4}\right)$ (d) $\frac{1}{k\pi} e^{k\pi^2/4}$

[TIFR-2023]

15. A surface is given by 4xy-2x+32-0
Which of the following is a vector normal to it at the point (2, 3, 1)
(a) $30\hat{\imath} - 8\hat{\jmath} + 9\hat{k}$ (b) $15\hat{\imath} - 4\hat{\jmath} + 9\hat{k}$
(c) $30\hat{\imath} + 8\hat{\jmath} - 9\hat{k}$ (d) $30\hat{\imath} - 8\hat{\jmath} - 9\hat{k}$

[TIFR-2023]

16. The value of the first derivative of the function $f(x) = \frac{2}{\sqrt{3}} e^{-\sqrt{3x^2|x|}}$
(a) 0 (b) 2
(c) $\frac{2}{\sqrt{3}}$ (d) Undefined

[TIFR 2024]

17. A surface is given by $2x^3z + 4y^2z + 3z^2 = 81$
Which of the following is a vector tangential to it at the point on the surface with coordinates (x, y, z) =$(1,2,3)$?
(a) $2i - 3j + 3k$ (b) $18i + 48j + 36k$
(c) $-3i + 2j + 6k$ (d) $-3i - 2j + 6k$

MATRICES

[TIFR-2010]

18. The matrix $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ can be related by a similarity transformation to the matrix
(a) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & -1 & 2 \end{pmatrix}$
(c) $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

[TIFR-2011]

19. Consider the matrix
$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$
A 3-dimensional basis formed by eigenvectors of M is

(a) $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ And $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ And $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

(c) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ And $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

(d) $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ And $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

[TIFR-2011]

20. The trace of the real 4×4 matrix $\mathbb{U} = \exp(\mathbb{A})$, where

$$\mathbb{A} = \begin{pmatrix} 0 & 0 & 0 & \frac{\pi}{4} \\ 0 & 0 & -\frac{\pi}{4} & 0 \\ 0 & \frac{\pi}{4} & 0 & 0 \\ -\frac{\pi}{4} & 0 & 0 & 0 \end{pmatrix}$$

Is equal to

(a) $2\sqrt{2}$

(b) $\pi/4$

(c) $\exp(i\varphi)\ for\ \varphi = 0, \pi$

(d) Zero

(e) $\pi/2$

(f) 2

[TIFR-2012]

21. The different 2×2 matrices A and B are found to have the same eigen-values. It is then correct to state that $A = SBS^{-1}$ where S can be a

(a) Traceless 2×2 matrix

(b) Hermitian 2×2 matrix

(c) Unitary 2×2 matrix

(d) Arbitrary 2×2 matrix

[TIFR-2014]

22. The product MN of two Hermitian matrices M and N is anti-Hermitian. It follows that

(a) $\{M, N\} = 0$ (b) $[M, N] = 0$

(c) $M^+ = N$ (d) $M^+ = N^{-1}$

[TIFR-2016]

23. If the eigenvalues of a symmetric 3×3 matrix A are 0, 1, 3 and the corresponding eigenvalues can be written as $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ respectively, then matrix A^4 is

(a) $\begin{pmatrix} 41 & -81 & 40 \\ -81 & 0 & -81 \\ 40 & 81 & 41 \end{pmatrix}$

(b) $\begin{pmatrix} -82 & -81 & 79 \\ -81 & 81 & -81 \\ 79 & -81 & 83 \end{pmatrix}$

(c) $\begin{pmatrix} 14 & -27 & 13 \\ -27 & 54 & -27 \\ 13 & -27 & 14 \end{pmatrix}$

(d) $\begin{pmatrix} 14 & -13 & 27 \\ -13 & 54 & -13 \\ 27 & -13 & 14 \end{pmatrix}$

[TIFR-2017]

24. The matrix

$$\begin{pmatrix} 100\sqrt{2} & x & 0 \\ -x & 0 & -x \\ 0 & x & 100\sqrt{2} \end{pmatrix}$$

where$x > 0$, is known to have two equal eigenvalues. Find the value of x.

[TIFR-2017]

25. A unitary matrix U is expanded in terms of a Hermitian matrix H, such that $U = e^{i\pi H/2}$

If we know that

$$H = \begin{pmatrix} 1/2 & 0 & \sqrt{3}/2 \\ 0 & 1 & 0 \\ \sqrt{3}/2 & 0 & -1/2 \end{pmatrix}$$

Then U must be

(a) $\begin{pmatrix} i & 1/2 & \sqrt{3}/2 \\ 1/2 & i & 1/2 \\ \sqrt{3}/2 & 1/2 & i \end{pmatrix}$

(b) $\begin{pmatrix} i/2 & 0 & \sqrt{3}/2 \\ 0 & 1 & 0 \\ i\sqrt{3}/2 & 0 & -i/2 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & \sqrt{3} \\ 0 & 2 & 0 \\ \sqrt{3} & 0 & -1 \end{pmatrix}$

(d) $\begin{pmatrix} 2i & 1 & \sqrt{3}/2 \\ 1 & 2i & 0 \\ \sqrt{3}/2 & 0 & 2i \end{pmatrix}$

[TIFR-2018]

26. If a 2×2 matrix M is given by

$$M = \begin{pmatrix} 1 & \frac{1-i}{\sqrt{2}} \\ \frac{1+i}{\sqrt{2}} & 0 \end{pmatrix}$$

Then $\det \exp M =$

(a) $e^2 e$ (b) e

(c) $2i \sin\sqrt{2}$ (d) $\exp(-2\sqrt{2})$

[TIFR-2019]

27. The eigenvalues of a 3×3 matrix M are

$\lambda_1 = 2\ \lambda_2 = -1\ \lambda_3 = 1$

And the eigenvectors are

$e_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $e_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ $e_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

The matrix M is

(a) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

[TIFR-2020]

28. The eigenvector e_1 corresponding to the smallest eigenvalue of the matrix $\begin{pmatrix} 2a^2 & a & 0 \\ a & 1 & a \\ 0 & a & 2a^2 \end{pmatrix}$. Where $a = \sqrt{\frac{3}{2}}$, is given (in terms of its transpose) by

(a) $e_1^T = \frac{1}{2}\left(\frac{1}{\sqrt{2}} - \sqrt{3}\frac{1}{\sqrt{2}}\right)$

(b) $e_1^T = \frac{1}{2}\left(\sqrt{\frac{3}{2}} \quad 1 \quad \sqrt{\frac{3}{2}}\right)$

(c) $e_1^T = \frac{1}{2}(1 \quad 0 \quad -1)$

(d) $e_1^T = \frac{1}{2}(1 \quad 0 \quad 1)$

[TIFR-2021]

29. A unitary matrix U is expressed in terms of a Hermitian matrix H such that if the j matrix H is given by $U = e^{i\pi H/2}$. If the matrix H is given by $H = \sqrt{3}\begin{pmatrix} 1/3 & 0 & \sqrt{2/3} \\ 0 & 1/\sqrt{3} & 0 \\ \sqrt{2/3} & 0 & -1/\sqrt{3} \end{pmatrix}$ then U will have the form

(a) $\begin{pmatrix} i/\sqrt{3} & 0 & i\sqrt{2}/\sqrt{3} \\ 0 & i & 0 \\ i\sqrt{2}/\sqrt{3} & 0 & -i/\sqrt{3} \end{pmatrix}$

(b) $\begin{pmatrix} \sqrt{3} & 0 & \sqrt{6} \\ 0 & 3\sqrt{3} & 0 \\ \sqrt{6} & 0 & -\sqrt{3} \end{pmatrix}$

(c) $\begin{pmatrix} i\sqrt{3} & 1\sqrt{3} & \sqrt{2}/\sqrt{3} \\ 1/\sqrt{3} & i & 1\sqrt{3} \\ \sqrt{2}/\sqrt{3} & 1\sqrt{3} & i\sqrt{3} \end{pmatrix}$

(d) $\begin{pmatrix} 3\sqrt{3i} & \sqrt{3} & 3/2 \\ \sqrt{i} & i & 0 \\ i\sqrt{2}/\sqrt{3} & 0 & 3\sqrt{3i} \end{pmatrix}$

[TIFR-2022]

30. Consider a set of three 3-dimensional vectors

$A = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $C = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

These vectors undergo a linear transformation

$A \to A' = \mathbb{M}A \qquad B \to B' = \mathbb{M}B \qquad C \to C' = \mathbb{M}C$

Where $\mathbb{M}$ is given by

$$\mathbb{M} = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

The volume of a parallelepiped whose sides are given by the transformed vectors A', B' and C' is

(a) 8 (b) 2

(c) 4 (d) 16

[TIFR-2023]

31. Consider a symmetric matrix

$M = \begin{pmatrix} \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 1 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \end{pmatrix}$

An orthogonal matrix (which can diagonalised this matrix by an orthogonal transformation

$O^T MO$ is given by $O =$

(a) $\begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}$

(b) $\begin{pmatrix} \sqrt{2/3} & 0 & \sqrt{1/3} \\ 0 & 1 & 0 \\ \sqrt{1/3} & 0 & -\sqrt{2/3} \end{pmatrix}$

(c) $\begin{pmatrix} \sqrt{1/3} & 0 & \sqrt{2/3} \\ 0 & 1 & 0 \\ \sqrt{2/3} & 0 & -\sqrt{1/3} \end{pmatrix}$

(d) $\begin{pmatrix} 1/\sqrt{2} & 0 & i/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -i/\sqrt{2} \end{pmatrix}$

[TIFR-2024]

32. Consider the following matrix

$$\mathbb{M} = \begin{pmatrix} 1 & 5 & -7 & 1 \\ 1 & 0 & 2 & 2 \\ 9 & -1 & 3 & 1 \\ 9 & 6 & -7 & -4 \end{pmatrix}$$

What is det $e^{\mathbb{M}}$?

(a) 1 (b) e

(c) e^{1210} (d) e^{-1210}

LINEAR ORDINARY DIFFERENTIAL EQUATIONS OF 1ST & 2ND ORDER

[TIFR-2013]

33. The differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

Has the complete solution, in terms of arbitrary constants A and B

(a) $A \exp x + B\, x \exp x$

(b) $A \exp x + B\, x \exp(-x)$

(c) $A\exp x + B\ \exp(-x)$

(d) $x\{A\exp x +\ B\exp(-x)\}$

[TIFR-2013]

34. Consider the surface corresponding to the equation $4x^2 + y^2 + z = 0$ A possible unit tangent to this surface at the point $(1,2,-8)$ is

(a) $\frac{1}{\sqrt{5}}\hat{\imath} - \frac{2}{\sqrt{5}}\hat{\jmath}$ (b) $\frac{4}{9}\hat{\imath} - \frac{8}{9}\hat{\jmath} + \frac{1}{9}\hat{k}$

(c) $\frac{1}{5}\hat{\jmath} - \frac{4}{5}\hat{k}$ (d) $-\frac{1}{\sqrt{5}}\hat{\imath} + \frac{3}{\sqrt{5}}\hat{\jmath} - \frac{4}{\sqrt{5}}\hat{k}$

[TIFR-2015]

35. Consider the differential equation

$\frac{d^2y}{dx^2} = -4\left(y + \frac{dy}{dx}\right)$

With the boundary condition that $y(x) = 0$ at $x = 1/5$. When plotted as a function of x, for $x \geq 0$, we can say with certainty that the value of y

(a) First increases, then decreases to zero

(b) First decreases, then increases to zero

(c) Has an extremum in the range $0 < x < 1$

(d) Oscillates from positive to negative with amplitude decreasing to zero

[TIFR-2016]

36. The function $y(x)$ satisfies the differential equation

$x\frac{dy}{dx} = y(In\, y - Inx + 1)$

With the initial condition $y(1) = 3$. What will be the value of $y(3)$?

[TIFR-2018]

37. If $y(x)$ satisfies differential equation $y'' - 4y' + 4y = 0$ with boundary conditions

$y(0) = 1$ and $y'(0) = 0$, then $y\left(-\frac{1}{2}\right) =$

(a) $\frac{2}{e}$ (b) $\frac{1}{2}\left(e + \frac{1}{e}\right)$

(c) $\frac{1}{e}$ (d) $\frac{e}{2}$

[TIFR-2019]

38. A set of polynomials of order n are given by the formula

$p_n(x) = (-1)^n \exp\left(\frac{x^2}{2}\right)\frac{d^n}{dx^n}\exp\left(-\frac{x^2}{2}\right)$

The polynomial $p_7(x)$ of order $n = 7$ is

(a) $x^7 - 21x^5 + 105x^3 - 105x - 21$

(b) $x^6 - 21x^5 + 105x^4 - 105x^3 + 21x^2 + x$

(c) $x^7 - 21x^5 + 105x^3 - 105x$

(d) $x^7 - 21x^5 + 105x^4 + 35x^3 - 105x$

[TIFR-2019]

39. The differential equation $x\frac{dy}{dx} - xy = \exp(x)$, where $y = e^2$ at $x = 1$, has the solution $y =$

(a) $\exp(x^2 + x)$

(b) $(1 - x)\exp(x) + \exp(1 + x)$

(c) $\exp(1 + x)\,(1 + In\, x)$

(d) $\exp(x)\ In\, x + \exp(1 + x)$

[TIFR-2020]

40. Consider the improper differential $đs = (1 + y^2)\,dx\, +\, xy\,dy$ An integrating factor for this is

(a) x (b) $1 + x^2$

(c) xy (d) $-1\, +\, y^2$

[TIFR-2021]

41. If y (x) satisfies the following differential equation $x\frac{dy}{dx} = \cot y - cosec\, y\cos x$ and we have $\lim_{x\to 0} y(x) = 0$ then $y(\pi/2)$

(a) $-\cos^{-1}\left(\frac{2}{\pi} - 2\right)$ (b) $\sin^{-1}(2/\pi)$

(c) $\pi/2$ (d) 0

[TIFR 2024]

42. Consider the following differential equations:

$\frac{dx}{dt} = \alpha y(t), \frac{dy}{dt} = a$ Where a is a positive constant. The solutions to these equations define a family of curves in the x, y plane. What are these curves?

(a) Parabolas (b) Circles

(c) Hyperbolas (d) Ellipses

FOURIER SERIES

[TIFR-2010]

43. A function $f(x)$ is defined in the range $-1 \leq x \leq 1$ by $f(x) = \begin{cases} 1 - x & for\ x \geq 0 \\ 1 + x & for\ x < 0 \end{cases}$. The first few terms in the Fourier series approximating this function are

(a) $\frac{1}{2} + \frac{4}{\pi^2}\cos\pi x + \frac{9}{9\pi^2}\cos 3\pi x + \cdots$

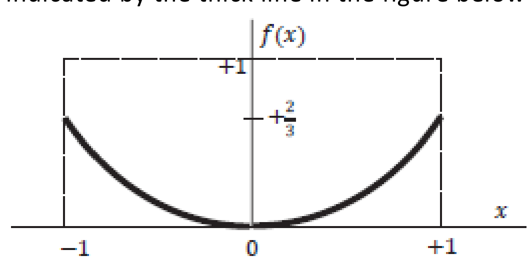
(b) $\frac{1}{2} + \frac{4}{\pi^2}\sin\pi x + \frac{9}{9\pi^2}\sin 3\pi x + \cdots$

(c) $\frac{4}{\pi^2}\cos\pi x + \frac{9}{9\pi^2}\cos 3\pi x + \cdots$

(d) $\frac{1}{2} - \frac{4}{\pi^2}\cos\pi x + \frac{9}{9\pi^2}\cos 3\pi x + \cdots$

[TIFR-2014]

44. A student is asked to find a series approximation for the function $f(x)$ in the domain $-1 \leq x \leq +1$, as indicated by the thick line in the figure below



The student represents the function by a sum of three terms

$$f(x) \approx a_0 + a_1 \cos\frac{\pi x}{2} + a_2 \sin\frac{\pi x}{2}$$

Which of the following would be the best choices for the coefficients a_0, a_1 and a_2?

(a) $a_0 = 1, a_1 = -\frac{1}{3}, a_2 = 0$

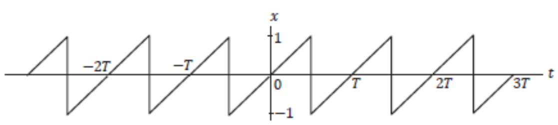
(b) $a_0 = \frac{2}{3}, a_1 = -\frac{2}{3}, a_2 = 0$

(c) $a_0 = \frac{2}{3}, a_1 = 0, a_2 = -\frac{2}{3}$

(d) $a_0 = -\frac{1}{3}, a_1 = 0, a_2 = -1$

[TIFR-2017]

45. Consider the waveform $x(t)$shown in the diagram below



The Fourier series for $x(t)$which gives the closest approximation to this waveform is

(a) $x(t) = \frac{2}{\pi}\left[cos\frac{\pi t}{T} - \frac{1}{2}cos\frac{4\pi t}{T} + \frac{1}{3}cos\frac{3\pi t}{T} + \cdots\right]$

(b) $x(t) = \frac{2}{\pi}\left[-sin\frac{\pi t}{T} + \frac{1}{2}sin\frac{4\pi t}{T} - \frac{1}{3}sin\frac{3\pi t}{T} + \cdots\right]$

(c) $x(t) = \frac{2}{\pi}\left[sin\frac{\pi t}{T} - \frac{1}{2}sin\frac{2\pi t}{T} + \frac{1}{3}sin\frac{3\pi t}{T} + \cdots\right]$

(d) $x(t) = \frac{2}{\pi}\left[-cos\frac{\pi t}{T} + \frac{1}{2}cos\frac{4\pi t}{T} - \frac{1}{3}cos\frac{6\pi t}{T} + \cdots\right]$

[TIFR-2020]

46. The sum of the infinite series $S = 1 + \frac{3}{5} + \frac{6}{25} + \frac{10}{125} + \frac{15}{625} + \ldots$

(a) $S = \frac{125}{64}$ (b) $S = \frac{25}{16}$

(c) $S = \frac{25}{24}$ (d) $S = \frac{16}{25}$

LAPLACE TRANSFORM

[TIFR-2016]

47. The integral $\int_0^\infty \frac{dx}{x}\left[\exp\left(-\frac{x}{\sqrt{3}}\right) - \exp\left(-\frac{x}{\sqrt{2}}\right)\right]$ evaluates to

(a) Zero (b) 2.03×10^{-2}

(c) 2.03×10^{-1} (d)2.03

COMPLEX ANALYSIS

[TIFR-2013]

48. If $|z| = x + iy$ then function $f(x, y) = (1 + x + y) + a(x^2 - y^2) - 1 + 2iy(1 - x - ax)$Where a is a real parameter, is analytic in the complex z plane if $a =$

(a) -1 (b) +1

(c) 0 (d) i

[TIFR-2016]

49. The value of the integral $\oint_C \frac{sinz}{z^6} dz$

Where C is the circle of center $z = 0$ and radius=1

(a) $i\pi$ (b) $i\pi/120$

(c) $i\pi/60$ (d)$-i\pi/6$

[TIFR-2017]

50. The value of the integral $\int_0^\infty \frac{dx}{x^4+4}$

Is

(a) π (b) $\frac{\pi}{2}$

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{8}$

[TIFR-2018]

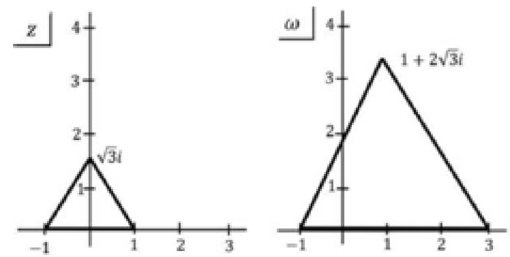
51. The value of the integral $\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}$ is

(a) 1/2a (b)$1/2\pi a$

(c) $\pi a e^{-a}$ (d) $\frac{e^{-a}}{a}$

[TIFR-2023]

52. A complex analytic function $\omega = f(z)$ transforms an equilateral triangle in the complex z- plane to another equilateral triangle in the complex ω-plane as shown in the figure.



Which of the options below can be $f(z)$?

(a) $f(z) = e^{\frac{5\pi i}{6}} z + 2i\sqrt{3}$

(b) $f(z) = 2e^{\frac{2\pi i}{3}} z + 2 + i\sqrt{3}$

(c)) $f(z) = 2e^{\frac{2\pi i}{3}} z + i\sqrt{3}$

(d) $f(z) = 2z + i\sqrt{3}$

[TIFR-2023]

53. Calculate the integral $\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)}$

(a) $\frac{\pi}{\sqrt{2}}$ (b) $\pi\sqrt{2}$

(c) 2π (d) π

TYLOR & LAURENT SERIES

[TIFR-2011]

54. The infinite series

$$x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots$$

Where$-1 < x < +1$, can be summed to the value

(a) $\tanh x$

(b) $\ln\left(1 - \frac{4}{\pi} tan^{-1} x\right)$

(c) $\frac{1}{2}\ln[(1+x)/(1-x)]$

(d) $\frac{1}{2}\ln[(1-x)/(1+x)]$

[TIFR-2017]

55. Evaluate the expression

$n!\int_0^A dx_{n-1}\int_0^{x_{n-1}} dx_{n-2}\int_0^{x_{n-2}} dx_{n-3}$

$\ldots.\int_0^{x_3} dx_2\int_0^{x_3} dx_1\int_0^{x_1} dx_0$

[TIFR-2024]

56. The following series

$$s=\sum_{n=1}^{\infty}(-1)^{1+n}\frac{1}{n2^{4n}}$$

Has the sum

(a) $S = In\left(\frac{17}{16}\right)$ (b) $s=\sqrt{\frac{17}{16}}$

(c) S is not convergent (d) $s=\frac{1}{1+\sqrt{\frac{1}{16}}}$

[TIFR-2024]

57. Let $F(\lambda)=\int_{-\infty}^{+\infty} dx\, e^{\lambda x-x^2}$

If the Taylor series expansion of $F(\lambda)$ around $\lambda=0$ $F(\lambda)=F_0+F_x\lambda+F_2\lambda^2+\cdots$.Then the value of F_2 is (You might find the following integral useful $\int_{-\infty}^{+\infty} ex\, e^{-ax^2}=\sqrt{\frac{\pi}{a}}\, for\ a>0)$

(a) $\sqrt{\pi}/4$ (b) $\sqrt{\pi}/8$

(c) $\sqrt{\pi}/2$ (d) $\sqrt{\pi}$

PROBABILITY

[TIFR-2010]

58. Consider a standard chess board with 8×8 squares. A piece starts from the lower left corner, which we shall call square (1, 1). A single move of this piece corresponds to either one step right, i.e. to square (1, 2) or one step forwards, i.e. to square (2, 1). If it continues to move according to these rules, the number of different paths by which the piece can reach the square (5, 5) starting from the square (1, 1) is

(a) 120 (b) 72

(c) 70 (d) 45

[TIFR-2011]

59. A 100 page book is known to have 200 printing errors distributed randomly through the pages. The probability that one of the pages will be found to be completely free of errors is closet to

(a) 67% (b) 50 %

(c) 25% (d) 13%

[TIFR-2014]

60. The probability function for a variable x which assumes only positive values is

$$f(x)=xexp\left(-\frac{x}{\lambda}\right)$$

Where $\lambda>0$. The ratio $\langle x\rangle/\hat{x}$ where $\hat{x}$ is the most probable value and $\langle x\rangle$ is the mean value of the variables$x$, is

(a) 2 (b) $\frac{1+\lambda}{1-\lambda}$

(c) $\frac{1}{\lambda}$ (d) 1

[TIFR-2015]

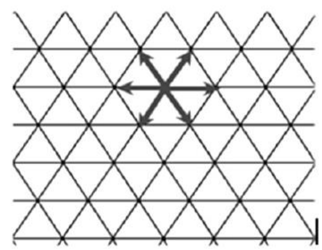
61. A random number generator outputs $+1$ or -1 with equal probability ever time it is run. After it is run 6 times, what is the probability that the sum of the answers generated is zero? Assume that the individual runs are independent of each other.

(a) 15/32 (b) 5/16

(c) 5/6 (d) ½

[TIFR-2016]

62. In a triangular lattice a particle moves from a lattice point to any of its 6 neighbouring points with equal probability, as shown in the figure on the right the probability that the particle is back at its starting point after 3 moves is



(a) 5/18 (b) 1/6

(c) 1/18 (d) 1/36

[TIFR-2021]

63. In a country, the fraction of population infected with Covid-19 is 0.2. It is also known that out of the people who are infected with Covid-19, only a fraction 0.3 show symptoms of the disease, while the rest do not show any symptoms. If you randomly select a citizen of this country, the probability that this person will NOT show symptoms of Covid-19 is

(a) 0.94 (b) 0.56

(c) 0.86 (d) 0.80

[TIFR-2023]

64. A faint star is known to emit light of a given frequency at an average rate of 10 photons per

minute. An astronomer plans to measure this rate using a photon-counting detector. If she wants to measure the rate to an accuracy of 5%, approximately how long should be the exposure time?

(a) 40 minutes (b) 1 hour
(c) 20 minutes (d) 10 minutes

[TIFR-2023]

65. A random positive variables follows an exponential distribution $p(x) \propto e^{-\lambda x}$. With $\lambda > 0$ The probability of observing an event $x > 3\langle x \rangle$, where $\langle x \rangle$ represents the average value of x Is

(a) $\frac{1}{e^3}$ (b) $\frac{1}{e^4}$
(c) $\frac{1}{e}$ (d) $\frac{1}{e^2}$

[TIFR-2024]

66. A student measures the radioactive decay of a material with a half-life of 13,000 years with a Geiger counter. In the laboratory notebook, the student records the following number of decays every 10 seconds:

158, 146, 145, 163, 154, 163, 160, 160, 152, 157, 154, 156, 149, 168, 152

The teacher suspects that the experiment was not done properly and the student created the numbers manually. Why would the teacher have such a suspicion?

(a) The variance is much less than the mean, unlike what is expected for a Poisson distribution.
(b) The standard deviation is much less than the variance, as expected for Poisson distribution.
(c) The median is less than the mean, unlike what is expected for a Poisson distribution.
(d) The median is greater than the mean, as expected for a Poisson distribution.

CURVE FITTING

[TIFR-2021]

67. Given the following x-y data table

X	1.0	2.0	3.0	4.0	5.0	6.0
y	0.602	0.984	1.315	1.615	1.894	2.157

Which would be the best-fit curve, where a and b are constant positive parameters?

(a) $y = bx^{1/(1+a)}$ (b) $y = ax - b$
(c) $y = a + e^{bx}$ (d) $y = a \log_{10} bx$

Answer Key									
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
(b)	(b)	(d)	(b)	(b)	(c)	(b)	1.07	18	(c)
11	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
(d)	(a)	10.64	(d)	(a)	(a)	(a)	(d)	(b)	(a)
21	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**
(c)	(a)	(c)	50	(b)	(b)	(d)	(a)	(a)	(a)
31	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**
(a)	(a)	(b)	(a)	(c)	81	(a)	(c)	(d)	(a)
41	**42**	**43**	**44**	**45**	**46**	**47**	**48**	**49**	**50**
(b)	(a)	(a)	(b)	(c)	(a)	(c)	(a)	(c)	(d)
51	**52**	**53**	**54**	**55**	**56**	**57**	**58**	**59**	**60**
(d)	(a)	(a)	(c)	A^n	(a)	(a)	(c)	(d)	(a)
61	**62**	**63**	**64**	**65**	**66**	**67**			
(b)	(c)	(a)	(a)	(a)	(a)	(c)			

SOLUTIONS

1. **Solution: (b)**

Here given function is step function and derivative of step function is delta function so option 2nd is correct.

2. **Solution: (b)**

$\int_0^{\infty} x^9 e^{-x^2} dx$

Let $x^2 = t \quad 2xdx = dt$

$I = \frac{1}{2}\int_0^{\infty} t^4 e^{-t} dt = \frac{\lceil 5}{2} = \frac{1}{2} \times 4 \times 3 \times 2 = 12$

3. **Solution: (d)**

$\delta(x^2 - \pi^2) = \frac{1}{|\pi - (-\pi)|}[\delta(x-\pi) + \delta(x+\pi)]$

$= \frac{1}{2\pi}[\delta(x-\pi) + \delta(x+\pi)$

$\therefore \int_{-\infty}^{\infty} dx \delta(x^2 - \pi^2)\cos x = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, [\delta(x-\pi) + \delta(x+\pi)]\cos x$

$= \frac{1}{2\pi}[\cos\pi + \cos(-\pi)] = \frac{1}{2\pi}(-1-1) = \frac{1}{\pi}$

Option (d) is correct

4. **Solution: (b)**

Let us begin with the relation

$\int f(\vec{r})\delta(\vec{r} - \vec{r})d\tau = f(\vec{r}) = f_1(r') = f_2(\theta')f_3(\Phi')$

Where A_1, A_2 and A_3 are to be found

$\iiint f(\vec{r})A_1\delta(r-r')A_2\delta(\theta-\theta')A_3\delta(\Phi - \Phi')r^2\sin\theta drd\theta d\Phi = f(r', \theta', \Phi')$

$\Rightarrow \int_0^{\infty} f_1(r)\delta(r-r')A_1 r^2 dr \int_0^{\pi} f_2(\theta)\sin\theta\delta(\theta - \theta')A_2 d\theta \int_0^{2\pi} f_3(\Phi)\delta(\Phi-\Phi')d\Phi = f_1(r)f_2(\theta)f_3(\theta)$

$\Rightarrow A_1 = \frac{1}{r^2}, A_2 = \frac{1}{\sin\theta,}, A_3 = 1$

So, $\delta(\vec{r} - \vec{r}) = \frac{1}{r^2}\delta(\vec{r} - \vec{r})\frac{1}{\sin\theta}\delta(\theta - \theta')\delta(\Phi - \Phi')$

Now, we can use a general formula

$\delta(f(x) = \sum_j \frac{1}{|-\sin\theta|}\delta(x - x_j)$

Where f is a function having only first order roots x_j

So, $\delta(\cos\theta - \cos\theta') = \frac{1}{|-\sin\theta|}\delta(\theta - \theta') = \frac{1}{\sin\theta}\delta(\theta - \theta')$

So, $\delta(\vec{r} - \vec{r}) = \frac{1}{r^2}\delta(\vec{r} - \vec{r})\delta(\cos\theta - \cos\theta')\,\delta(\Phi - \Phi')$

5. **Solution: (b)**

Let $F(x) = x - \int_0^x f(t).dt$

$\frac{d}{dx}f(x) = 1 - f(x)$

$\frac{d}{dx}f(x) + f(x) = 1$

Integrating factor $= e^{\int dx} = e^x$

Solution of above equation is \

$f(x)e^x = \int e^x.dx$

$f(x)e^x = e^x + c$

$f(x) = 1 + ce^{-x}$.... (1)

Consider at $x = 0$

$f(x=0) = 0$

$\therefore$ Put in equation (1)

$= 0 = 1 + c \Rightarrow c = -1$

$\therefore f(x) = 1 - e^{-x}$

If x is in. $f(x)$ is also

$\therefore$ Option (b) is correct

6. Solution: (c)

Vector perpendicular to the surface $x^2y + 2xz = 4$ at the point $(2, -2, 3)$ lies in direction of gradient ∇S

$\nabla S = (2xy + 2z)\hat{\imath} + x^2 j + 2xk$

$\nabla S|_{(2,-2,3)} = -2\hat{\imath} + 4j + 4k$

Vector parallel to surface will be normal to ∇S we have to check each option by taking dot product with ∇S we find that $6\hat{\imath} - 2j + 5k$ is perpendicular to ∇S.

7. Solution: (b)

$$\langle \cos 4x \rangle = \frac{\int_0^\infty \cos 4x f(x) dx}{\int_0^\infty f(x) dx}$$

$$\frac{\int_0^\infty \cos 4x e^{-4x} dx}{\int_0^\infty e^{-4x} dx}$$

$$I = \int_0^\infty \cos 4x e^{-4x} dx$$

$= \frac{e^{-4x}}{4}\sin 4x + \int e^{-4x} \sin 4x\, dx$ (Integration by parts)

$= 2I = \frac{e^{-4x}}{4}(\sin 4x - \cos 4x)$

$I = \frac{e^{-4x}}{8}(\sin 4x - \cos 4x)|_0^\infty = \frac{1}{8}$

$\int_0^\infty d^{-4x} dx = -\frac{\frac{e^{-4x}}{4}|_0^\infty 1}{4}$

$\langle \text{Cos } 4x \rangle == \frac{1}{8} - \frac{1}{4} = \frac{1}{2}$

8. Solution: (1.07)

$\int_{-\infty}^\infty cos(\sqrt{2x}) = 2\int_0^\propto e^{-x^2} cos\sqrt{2x}\, dx$

Consider integral $\int_{-\infty}^\infty e^{-x^2 dz = \int_C f(z)dz}$

Where C is the rectangle with vertices at the points $O, R, R + ia$ and ia. Since, $f(z)$ has no poles within and on this contour, we have by Cauchy's theorem

$\int_C f(z)dz = \int_0^R f(x)dx + \int_0^R f(R + iy)dx + \int_R^0 f(x + ia)dx + \int_a^0 f(iy) i\, dy$

$= I_1 + I_2 + I_3 + I_4$

$|I_2| = \left|\int_0^a exp(-(R + iy)^2) i\, dy\right|$

$\leq \int_0^a |exp(-R^2 + y^2 - 2iRy) i| dy$

$\leq \int_0^a e^{-R^2 e^{y^2 dy}} \to 0\ as\ R \to \infty$

$Hence, when\ R \to \infty, we\ get\ from\ equation\ (1)$

$\int_0^\infty e^{-x^2} dx + 0\int_\infty^0 exp\{-(x + ia)^2\}dx - i\int_0^\infty e^{y^2} dy = 0$

$\sqrt{\frac{\pi}{2}} - e^{a^2}\int_0^\infty e^{-x^2}(cos\ 2ax - i\ sin\ 2ax)dx - i\int_0^\infty e^{y^2} dy = 0$

$e^{a^2}\int_0^\infty e^{-x^2}(cos 2ax - isin\ 2\ ax)dx = \sqrt{\frac{\pi}{2}} - i\int_0^\infty e^{y^2} dy = 0$

Equating real part

$e^{a^2}\int_0^\infty e^{-x^2} cos 2ax\ dx = \sqrt{\frac{\pi}{2}}$

Or,

$\int_0^\infty e^{-x^2} cos\ 2ax = \sqrt{\frac{\pi}{2}} e^{-a^2}$

Put $a = \frac{1}{\sqrt{2}}$

$\int_{-\infty}^\infty e^{-x^2} cos\sqrt{x}\ dx = 2\int_0^\infty e^{-x^2} cos\sqrt{x}\, dx = \sqrt{\pi} e^{-\frac{1}{2}} = \sqrt{\frac{\pi}{e}} = 1.07$

9. Solution: (18)

The three condition (i), (ii), (iii) impose symmetries on the tensor. Let's analyze them

(i) $T_{ijk\ell} = T_{ijk\ell}$ Indicator that the tensor is symmetric under the exchange of indices i and j

(ii) $T_{ijk\ell} = T_{ijk\ell}$ Indicates that tensor is symmetric under the exchange of indices k and l

(iii) $T_{ijk\ell} = T_{ijk\ell}$ Indicates that tensor is symmetric under the exchange of indices i and j, k and l

For a symmetric tensor the number of independent components.

For the indices I and j. We have $3 \times 3 = 9$ components. However, due to the symmetry under the exchange of I and J, we only need to consider the unique combinations, which gives us $\frac{1}{2} \times 9 = 4.5$

Similarly, for the indices k and l we have $3 \times 3 = 9$ component again, due to the symmetric under the exchange of k and ℓ we only need to consider the unique combinations,

Which gives us $\frac{1}{2} \times 9 = 4.5$

Finally for the indices i, j, k, l we have $3 \times 3 \times 3 \times 3 = 81$ components

Due to the symmetries (iii) we need to consider the combinations, which gives as $\frac{1}{31} \times 81 = 9$

So, dthe total number of independent component is

$$4.5 + 4.5 + 9 = 18$$

Therefore, there are 18 independent components of the fourth-rank Cartesian tensor $T_{ijk\ell}$ satisfying the given conditions.

10. Solution: - (c)

Equation of surface is given by

$S = ax^2 + by^2 + cz + d = 0$

$\nabla S - 2a\hat{\iota} + 2\hat{bj} + c\hat{k}$

$\nabla S|_{(1,1,0)} = 2a\hat{\imath} + 2\hat{bj} + \widehat{ck}$

$\nabla S|_{(0,0,1)} = \widehat{ck}$

$\widehat{n_1} = \frac{\nabla S}{|\nabla S|} = \frac{a\hat{\imath}+2\widehat{bj}+c\hat{k}}{\sqrt{4a^2+4b^2+c^2}}$

$\widehat{n_2} = \frac{\nabla S}{|\nabla S|} = \hat{k}$

Unit vector $\widehat{m}$ will lie in direction of $\widehat{n_1} \times \widehat{n_2}$

$\widehat{n_1} \times \widehat{n_2} = \frac{a\hat{\imath}+2\widehat{bj}+c\hat{k}}{\sqrt{4a^2+4b^2+c^2}} \times \hat{k}$

$= \frac{-2a\hat{\jmath}+2\widehat{bi}}{\sqrt{4a^2+4b^2+c^2}}$

Unit vector along $\widehat{n_1} \times \widehat{n_2}$, $\widehat{m}\frac{a\hat{\imath}+2\widehat{bj}}{\sqrt{(a^2+b^2)}}$

11. Solution: (d)

The given array $T_{ijk\ell}$ is a fourth -rank Cartesian tensor with indices i, j, k, l each taking values in the range 1 to 4.

The symmetry condition $T_{ijkl} = T_{ijlk} = T_{klij} = -T_{klij}$ indicator that the tensor is completely antisymmetric with respect to the exchange of any pair of Indices'. In an antisymmetric tensor, swapping any two indices leads to a sign change,

For a completely antisymmetric fourth-rank tensor in 4 dimensions, the number of independent components can be found using the formula.

$\therefore$ Number of independent components $= \frac{1}{4!} \times (4!)^2 = 1$

$\therefore$ Option (d) is correct.

12. Solution: (a)

$x \log\frac{x+1}{x-1} = x \log\frac{x+1/x}{x-1/x}$

$= x\left[\log\left(1+\frac{1}{x}\right) - \log\left(1-\frac{1}{x}\right)\right]$

$= x\left[\left(\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^2} \dots\right) - \left(-\frac{1}{x} - \frac{1}{2}x^2 - \frac{1}{3x^2} \dots\right]\right.$

$= x\left[\frac{2}{x} + \frac{2}{3x^3} \dots .\right]$

$= \left[2 + \frac{2}{3x^2} + \cdots\right]$

So $\lim_{x\to\infty} x \log\frac{x+1}{x-1} = 2$

13. Solution: (10.64)

Let $I = \int_0^{\pi/2} \sin^8\theta \cos^{11}\theta \, d\theta$

We have

$\int_0^{\pi/2} \sin^8\theta \cos^9\theta \, d\theta = \frac{\left|\frac{P+1}{2}\right|\frac{q+1}{2}}{2\left|\frac{P+q+2}{2}\right.}$

$\therefore \int_0^{\pi} \sin^8\theta \cos^{11}\theta \, d\theta = \frac{\left|\frac{8+1}{2}\right|\frac{11+1}{2}}{2\left|\frac{11+8+1}{2}\right.} = \frac{\left|\frac{9}{2}\right|\frac{12}{2}}{2\left|\frac{21}{2}\right.}$

$= \frac{120\times 26}{130945815}$

$\therefore \int_0^{\pi/2} \sin^8\theta \cos^{11}\theta \, d\theta = 5.8650 \times 10^{-5}$

$\int_0^{\infty} r^{19}\, e^{-r^2} dr = \frac{1}{2}\frac{\left|\frac{19+1}{2}\right.}{(2)^{10}} = \frac{1}{2} \times 9! = 181440$

$\therefore I = \int dr\, d\theta\, r^{19}\, e^{-r^2} \sin^8\theta \cos^{11}\theta$

$= 5.8650 \times 10^{-5} \times 18440$

$p = 10.6414$

14. Solution: (d)

$I = \int_{-\frac{\pi}{2}}^{\pi/2} dx \cos h\, kx^2 \sin^2 x.dx =$

$2\int_0^{\frac{\pi}{2}} dx \cos h\, kx^2 \sin^2 x.dx$

It is given k is very large so $\cos kx^2 = \frac{\exp kx^2}{2}$

The integration

$$I = 2\int_0^{\pi/2} dx\, \frac{\exp kx^2}{kx} \sin^2 x.dx$$

$$= \int_0^{\frac{\pi}{2}} dx\; 2\, kx \exp kx^2 \left(\frac{\sin^2 x}{2kx}\right).dx$$

$$= \int_0^{\frac{\pi}{2}} dx\; 2kx \exp kx^2 \left(\frac{\sin^2 x}{2kx}\right).dx$$

Using integration by parts

$I = \frac{\exp kx^2}{k}\left(\frac{\sin x}{x}\right)\sin x\; |_0^{\frac{\pi}{2}} -$

$\int_0^{\pi/2} \frac{d}{dx}\left(\frac{\sin^2 x}{kx}\right) \exp kx^2.dx]$

$I = \frac{1}{k\pi} \exp kx^2 dx$ will contain $\frac{1}{k^2}$ term which will be zero at large k

$$\therefore I = \frac{1}{k\pi} \exp\left(\frac{k\pi^2}{4}\right)$$

$\therefore$ Option (4) is correct

15. Solution: (a)

$\nabla f(x,y,z)_{(2,3,1)} \Rightarrow \frac{\partial f}{\partial x}\,\iota + \frac{\partial f}{\partial y}\grave{j} + \frac{\partial f}{\partial z}\hat{k}\Big|_{(2,3,1)}$

$= (8xy - 2y^2)\hat{\imath} + (4x^2 - 4xy)j + 9z^2k$

$= (8xy - 2y^2)\hat{\imath} + (4x^2 - 4xy)\hat{\jmath} + 9Z^2\grave{k}\Big|_{(2,3,1)} =$

$[(8xy - 2y^2)\hat{\imath} + (4x^2 - 4xy)\hat{\jmath} + 9z^2\grave{k}]\Big|_{(2,3,1)}$

$= 30\hat{\imath} - 8\hat{\jmath} + 9\grave{k}$

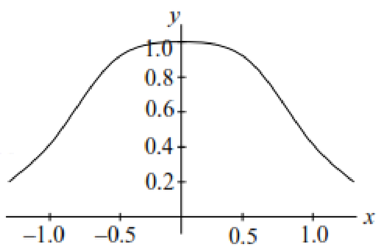
16. Solution: (a)

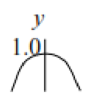
$f(x) = \frac{2}{\sqrt{3}}e^{-\sqrt{3x^2|x|}} = \begin{cases} \frac{2}{\sqrt{3}}e^{-\sqrt{3x^3}} \\ \frac{2}{\sqrt{3}}e^{\sqrt{3x^3}} \end{cases}$

$f'(x)|_{x=0^+} = \frac{2}{\sqrt{3}} \times 3x^2\sqrt{3e^{-\sqrt{3x^3}}}\Big|_{x=0} = 0$

$f'(x)|_{x=0^-} = \frac{2}{\sqrt{3}} \times 3x^2\sqrt{3e^{\sqrt{3x^3}}}\Big|_{x=0} = 0$

Thus $f'(0) = 0$





(x from -1.2 to 1.2)

17. Solution: (a)

$2x^3z + 4y^2z + 3z^2 = 81 \qquad given$

$\nabla\phi = 6x^2z\hat{\imath} + 8yz\hat{\jmath} + (2x^3 4y^2 + 6z)\hat{k}$

$\nabla\phi|_{(1,2,3)} = (6 \times 1 \times 3)\hat{\imath} + (8 \times 2 \times 3)\hat{\jmath}(2 \times 4 \times 4 + 18)\hat{k}$=$78\hat{\imath} + 48\hat{\jmath} + (20 + 16)\hat{k}$

$\nabla\phi|_{(1,2,3)} = 18\hat{\imath} + 48\hat{\jmath} + 36\hat{k}$

The vector tangential to surface should be

$\perp er\ to\ \nabla\phi|_{(1,2,3)}$

$\therefore (18\hat{\imath} + 48\hat{\jmath} + 36\hat{k}).(2\hat{\imath} - 3\hat{\jmath} + 3\hat{k})$

$= 36 - 144 + 108 = 0.$

Option (a) is correct.

18. Solution: (d)

Here

$\det\begin{bmatrix}1 & 1 & 0\\1 & 1 & 1\\0 & 1 & 1\end{bmatrix} = -1$

Trance of $\begin{bmatrix}1 & 1 & 0\\1 & 1 & 1\\0 & 1 & 1\end{bmatrix} = 3$

Only (d) gives determinant -1 and trace 3.

Similarly matrices have same eigenvalues and hence, then determinant and trace are same.

19. Solution: (b)

$M = \begin{bmatrix}1 & 0 & 0\\0 & 0 & -1\\0 & -1 & 0\end{bmatrix}$

$|M - \lambda I| = 0$

$\begin{vmatrix}1-\lambda & 0 & 0\\0 & -\lambda & -1\\0 & -1 & -\lambda\end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(\lambda^2 - 1) = 0$

$\Rightarrow \lambda = 1, 1, -1$

For $\lambda = -1, [A - \lambda I]X = 0$

$\Rightarrow \begin{bmatrix}2 & 0 & 0\\0 & 2 & -1\\0 & -1 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$2x_1 = 0$

$\Rightarrow x_2 - x_3 = 0$

$x_2 = x_3$

So, eigenvector corresponding to $\lambda = -1$ is $\begin{bmatrix}0\\1\\1\end{bmatrix}$

For $\lambda = 1$

$[M - \lambda I] = 0$

$\begin{bmatrix}0 & 0 & 0\\0 & -1 & -1\\0 & -1 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$x_2 + x_3 = 0$

$x_2 = -x_3$

Eigenvectors corresponding to $\lambda = 1$ is given by

$\begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}$

20. Solution: (a)

$A = \begin{bmatrix}0 & 0 & 0 & \frac{\pi}{4}\\0 & 0 & -\frac{\pi}{4} & 0\\0 & \frac{\pi}{4} & 0 & 0\\-\frac{\pi}{4} & 0 & 0 & 0\end{bmatrix}$

$= \frac{\pi}{4}\begin{bmatrix}0 & 0 & 0 & 1\\0 & 0 & -1 & 0\\0 & 1 & 0 & 0\\-1 & 0 & 0 & 0\end{bmatrix}$

$A^2 = \frac{\pi^2}{16}\begin{bmatrix}0 & 0 & 0 & 1\\0 & 0 & -1 & 0\\0 & 1 & 0 & 0\\-1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 0 & 1\\0 & 0 & -1 & 0\\0 & 1 & 0 & 0\\-1 & 0 & 0 & 0\end{bmatrix}$

$= \frac{\pi^2}{16}\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

$A^3 = -\frac{\pi^2}{64}A$

$A^4 = -\frac{\pi^4}{256}I$

Now, $e^4 = I + 4 + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \cdots$

$= I + A - \frac{1}{2!}\left(\frac{\pi}{4}\right)^2 I - \frac{1}{3!}\left(\frac{\pi}{4}\right)^2 A + \frac{1}{4!}\left(\frac{\pi}{4}\right)^4 I + \cdots \ldots$

Trace of $A = 0$ and Trace of $I = 4$

Trace $e^4 = 4\left[1 - \frac{1}{2!}\left(\frac{\pi}{4}\right)^3 + \frac{1}{4!}\left(\frac{\pi}{4}\right)^4 - \cdots\right]$

$= 4\cos\frac{\pi}{4} = 2\sqrt{2}$

21. Solution: (c)

Here S will diagonalised B. S form by arranging all eigenvectors of B in columns of matrix and it follows $S^\dagger = S^{-1}$. Thus S is unitary 2×2 matrix.

22. Solution: (a)

$(MN)^\theta = -MN$

$N^\theta M^\theta = -MN$

$\Rightarrow NM = -MN$

So, $MN + NM = 0$

$\{M, N\} = 0$

23. Solution: (c)

Let $s = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}$

And $\Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

$S^{-1} = \frac{1}{6}\begin{bmatrix} -2 & -2 & -2 \\ -3 & 0 & 3 \\ -1 & 2 & -1 \end{bmatrix}$

$As = s\Lambda$

$A = S\Lambda S^{-1}$

$\Rightarrow A^{-4}(S\Lambda S^{-1})^4 = S\Lambda^4 S^{-1}$

$= -\frac{1}{6}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 81 \end{bmatrix}\begin{bmatrix} -2 & -2 & -2 \\ -3 & 0 & 3 \\ -1 & 2 & -1 \end{bmatrix}$

$= -\frac{1}{6}\begin{bmatrix} 0 & 1 & 81 \\ 0 & 0 & -162 \\ 0 & -1 & -81 \end{bmatrix}\begin{bmatrix} -2 & -2 & -2 \\ -3 & 0 & 3 \\ -1 & 2 & -2 \end{bmatrix}$

$= -\frac{1}{6}\begin{bmatrix} -84 & 162 & -78 \\ 162 & -324 & 162 \\ -78 & 162 & -84 \end{bmatrix} = \begin{bmatrix} 14 & -27 & 13 \\ -27 & 54 & -27 \\ 13 & -27 & 14 \end{bmatrix}$

24. Solution: (50)

The given matrix is

$= \begin{bmatrix} 100\sqrt{2} & x & 0 \\ -x & 0 & -x \\ 0 & x & 100\sqrt{2} \end{bmatrix}$

We can notice that one of the eigenvectors is $X = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ with corresponding eigenvalues $\lambda = 100\sqrt{2}$ as

$AX = 100\sqrt{2}\,X$

So, one of eigenvalue, $\lambda = 100\sqrt{2}$

Sum of eigenvalues = trace of matrix

$\lambda_1 + \lambda_2 + \lambda_3 = 200\sqrt{2}$

Products of eigenvalues = determinant of matrix

$\lambda_1\lambda_2\lambda_3 = 200\sqrt{2}x^2$

$\lambda_1 = 100\sqrt{2}$ cannot be repeated two time because in that case one of the eigenvalues will be zero but determinant is non zero. So, $\lambda_1 = \lambda_3$

$\lambda_1 + \lambda_2 + \lambda_3 = \lambda_1 + 2\lambda_2 = 200\sqrt{2}$

$\Rightarrow \lambda_2 = \frac{(200\sqrt{2} - 100\sqrt{2})}{2} = 50\sqrt{2}$

$\lambda_2 = \lambda_3 = 50\sqrt{2}$

$\lambda_1 + \lambda_2 + \lambda_3 = 200\sqrt{2}x^2$

$\Rightarrow 200\sqrt{2}x^2 = 100\sqrt{2} \times 50\sqrt{2} \times 50\sqrt{2}$

$x^2 = 50 \times 50$

$\Rightarrow x = 50$

25. Solution: (b)

$H = \begin{bmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{bmatrix}$

H is a rotational matrix about y-axis

One of the eigenvalue is 1. Let the other eigenvalues are λ_2 and λ_3

Sum of eigenvalues, $1 + \lambda_2 + \lambda_3 = 1$

$\Rightarrow \lambda_2 + \lambda_3 = 0 \Rightarrow \lambda_3 = -\lambda_2$

Product of eigenvalues are $1, 1, -1$

Let us find eigenvector

For $\lambda = 1, (H - \lambda I)X = 0$

$\begin{bmatrix} -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & -\frac{3}{2} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\sim \begin{bmatrix} -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$X_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} \sqrt{3} \\ 0 \\ 1 \end{bmatrix}, X_2' = \begin{bmatrix} \sqrt{3} \\ 0 \\ \frac{1}{2} \end{bmatrix}$ after normalization

For $\lambda = -1, (H - \lambda I)X = 0$

$$\begin{bmatrix} \frac{3}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 2 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{3}{2} & 0 & \frac{1}{2} \\ \frac{3}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_2 \leftrightarrow R_3, R_2 \leftrightarrow R_1 \setminus$$

$$\sim \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ 0 \\ 1 \end{bmatrix}, X_3' = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{\sqrt{3}}{2} \end{bmatrix} \text{ after normalization}$$

Now, using diagonalization

$HS = S\Lambda$

$H = A\Lambda S^{-1}$

$$\text{When } S = \begin{bmatrix} 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 1 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$\Delta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

So, $U = e^{igH/2}$

$$= S\begin{bmatrix} e^{-\frac{i\pi\lambda_1}{2}} & 0 & 0 \\ 0 & e^{-\frac{i\pi\lambda_2}{2}} & 0 \\ 0 & 0 & e^{-\frac{i\pi\lambda_3}{2}} \end{bmatrix}S^{-1}$$

$$= \begin{bmatrix} 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 1 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} e^{\frac{i\pi}{2}} & 0 & 0 \\ 0 & e^{\frac{i\pi}{2}} & 0 \\ 0 & 0 & e^{\frac{i\pi}{2}} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{\sqrt{3}i}{2} & \frac{i}{2} \\ i & 0 & 0 \\ 0 & \frac{i}{2} & -\frac{\sqrt{3}i}{2} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{i}{2} & 0 & \frac{\sqrt{3}i}{2} \\ 0 & i & 0 \\ \frac{\sqrt{3}i}{2} & 0 & -\frac{i}{2} \end{bmatrix}$$

The solution is lengthy, we can get the answer by simply observing the option. Out of the four option, only the matrix in () is unitary. So (b) is the obvious answer.

26. Solution: (b)

$$M = \begin{bmatrix} 1 & \frac{1-i}{\sqrt{2}} \\ \frac{1+i}{\sqrt{2}} & 0 \end{bmatrix}$$

Let, λ_1 & λ_2 are eigenvalues of $\mathbb{M}$

So, eigenvalues of $\exp M$ will be e^{λ_1} and e^{λ_2} respectively $det(\exp \mathbb{M})$=product of eigenvalues of $\exp M = e^{\lambda_1}.e^{\lambda_2} = e^{trace(M)}$=$e^1 = e$

27. Solution: (d)

$\widehat{e_2} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is not eigenvectors of matrix in (b) & (c).

$\widehat{e_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is not eigenvector of matrix in (d).

So, only option left is (a) and we can clearly verify that $\widehat{e_1}$, $\widehat{e_2}$ and $\widehat{e_3}$, are eigenvector of matrix in (d).

28. Solution: (a)

Here Let given matrix be A

$$A = \begin{bmatrix} 2a^2 & a & 0 \\ a & 1 & a \\ 0 & a & 2a^2 \end{bmatrix} \text{ Where } a = \sqrt{\frac{3}{2}}$$

$$= \begin{bmatrix} 3 & \sqrt{\frac{3}{2}} & 0 \\ \sqrt{\frac{3}{2}} & 1 & \sqrt{\frac{3}{2}} \\ 0 & \sqrt{\frac{3}{2}} & 3 \end{bmatrix}$$

(a) $$Ae_1 = \begin{bmatrix} 3 & \sqrt{\frac{3}{2}} & 0 \\ \sqrt{\frac{3}{2}} & 1 & \sqrt{\frac{3}{2}} \\ 0 & \sqrt{\frac{3}{2}} & 3 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\sqrt{3} \\ \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So, $e_1$is eigenvector corresponding to $\lambda = 0$

(b) $$Ae_1 = \frac{1}{2}\begin{bmatrix} 3 & \sqrt{3/2} & 0 \\ \sqrt{3/2} & 1 & \sqrt{3/2} \\ 0 & \sqrt{3/2} & 3 \end{bmatrix}\begin{bmatrix} \sqrt{3/2} \\ 1 \\ \sqrt{3/2} \end{bmatrix} =$$

$$\frac{1}{2}\begin{bmatrix} 4\sqrt{3/2} \\ 4 \\ 4\sqrt{3/2} \end{bmatrix} = 2\begin{bmatrix} \sqrt{3/2} \\ 1 \\ \sqrt{3/2} \end{bmatrix}$$

So, $e_1$is eigenvector corresponding to $\lambda = 2$

(c) $$Ae_1 = \frac{1}{2}\begin{bmatrix} 3 & \sqrt{3/2} & 0 \\ \sqrt{3/2} & 1 & \sqrt{3/2} \\ 0 & \sqrt{3/2} & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix} =$$

$$3\begin{bmatrix} \sqrt{1/2} \\ 1 \\ -\sqrt{1/2} \end{bmatrix}$$

So, $e_1$is eigenvector corresponding to $\lambda = 3$

(d) e_1 is not eigenvector

29. Solution: (a)

Given, $H = \sqrt{3}\begin{bmatrix} \frac{1}{3} & 0 & \sqrt{2/3} \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ \sqrt{\frac{2}{3}} & 0 & -\frac{1}{\sqrt{3}} \end{bmatrix}$

For unitary matrix, every Row and column is orthonormal…

Let check for each options:

a) $|R_1| = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + 0^2 + \left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2} = \sqrt{\frac{1+2}{3}} = \sqrt{\frac{3}{3}} = 1$

$|R_2| = \sqrt{0+1+0} = 1$

$|R_3| = \sqrt{\frac{2}{3} + 0 + \frac{1}{3}} = \sqrt{\frac{3}{3}} = 1$

From option (b)

$|R_1| = \sqrt{3+0+6} = \sqrt{9} = 3 \neq 1$

So option (b) is not correct

For option(c)

$$|R_1| = \sqrt{\left(\sqrt{3}\right)^2 + \left(\sqrt{3}\right)^2 + \left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2} = \sqrt{3+3+\frac{2}{3}}$$

$$= \sqrt{\frac{8}{3}} \neq 1$$

So option (c) is not correct

For option (d)

$$|R_1| = \sqrt{3\left(\sqrt{3}\right)^2 + \left(\sqrt{3}\right)^2 + \left(\frac{3}{2}\right)^2} = \sqrt{27+3+\frac{9}{4}}$$

$$\neq 1$$

∴ In option (a) all rows are arthonormal

∴Option (a) is correct

30. Solution: (a)

A set of three-3- dimensional vectors is given by

$A = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

These vectors undergo a linear transformation

$A \to A' = MA, B \to B' = MB, C \to C' = MC$

M is given by

$$M = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$$A' = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$B' = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$C' = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 10 \\ 3 \\ 5 \end{pmatrix}$$

The volume of a parellopiped whose sides are given by the transformed vectors $A', B', \& C'$ is

Det $\begin{pmatrix} 1 & 1 & 10 \\ 1 & 0 & 3 \\ 2 & 1 & 5 \end{pmatrix} = 8$

Option (a) is correct

31. Solution: (a)

$$\begin{vmatrix} \frac{1}{3}-\lambda & 0 & \frac{1}{3} \\ 0 & 1-\lambda & 0 \\ \frac{2}{3} & 0 & \frac{1}{3}-\lambda \end{vmatrix} = 0$$

$$\left(1-\lambda\left[\left(\frac{1}{3}-\lambda\right)^2 - \frac{4}{9}\right] = 0\right)$$

$$\left(1-\lambda\left[\left(\frac{1}{3}-\lambda\right)^2 - \frac{4}{9}\right] = 0\right)$$

∴ Eigen vector

for $\lambda = 1$

$\begin{pmatrix} k_1 \\ y \\ k_1 \end{pmatrix}$

for $k_1 = \frac{1}{\sqrt{2}}, y = 0, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

for $k_1 = 0, y = 1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

For $\lambda = 0 - \frac{1}{3}$

$$\begin{pmatrix} 2/3 & 0 & 2/3 \\ 0 & 4/3 & 0 \\ 2/3 & 0 & 2/3 \end{pmatrix} = \begin{pmatrix} x \\ z \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x = -z$ and $y = 0$

Eigenvector for $\lambda = \frac{1}{3}$ is $\begin{pmatrix} 1\sqrt{2} \\ 0 \\ -1\sqrt{2} \end{pmatrix}$

Thus $0 = \begin{pmatrix} 1\sqrt{2} & 0 & 1\sqrt{2} \\ 0 & 1 & 0 \\ 1\sqrt{2} & 0 & -1\sqrt{2} \end{pmatrix}$

32. Solution: (a)

$$M = \begin{pmatrix} 1 & 5 & -7 & 1 \\ 1 & 0 & 2 & 2 \\ 9 & -1 & 3 & 1 \\ 9 & 6 & -7 & -4 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & 5 & -71 \\ 1 & 0 & 2\ 2 \\ 9 & -1 & 3\ 1 \end{pmatrix}$$

Since $\det(e^m) = e^{trace}(M)$

Since trace $(M) = 0$

$$\det(e^M) = 1$$

33. Solution: (b)

$$(D^2 - 2D + 1)y = 0$$

$$y = (c_1 + c_2 x)e^x$$

34. Solution: (a)

Normal to surface $4x^2 + y^2 + z = 0$ lies in the direction of ∇S

$$\nabla S = 8x\hat{\imath} + 2y\hat{\jmath} + \hat{k}|_{(1,2,-8)}$$

$$8x\hat{\imath} + 2y\hat{\jmath} + \hat{k}$$

The possible tangent will be perpendicular to ∇S

$$\left(\frac{1}{\sqrt{5}}\hat{\imath} - \frac{2}{\sqrt{5}}\hat{\jmath}\right) \perp \nabla S$$

35. Solution: (c)

$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$

$(D^2 + 4D + 4)y = 0$

At $y = (c_1 + c_2 x)e^{-2x}$

$x = 1/5, y = 0$

$\Rightarrow c_1 + \frac{1}{5}c_2 = 0 \Rightarrow c_2 = -5c_1$

$y(x) = c_1[-2(1 - 5x) - 5]e^{-2x}$

$\frac{dy}{dx} = 0$, at $x = 0.7$

So, $y(x)$ must have extremum at $x = 0.7$

36. Solution: (81)

$x\frac{dy}{dx} = y(In\, y - In\, x + 1)$

$\frac{1}{y}\frac{dy}{dx} = \frac{In\, y - In\, x + 1}{x}$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} - \frac{1}{x}In\, y = \frac{1 - In\, x}{x}$

Let $In\, y = z$

$\Rightarrow \frac{dz}{dx} = \frac{1}{y}.\frac{dy}{dx}$

$\Rightarrow \frac{dz}{dx} - \frac{z}{x} = \frac{1 - In\, x}{x}$

I.F. $= e^{\int \frac{1}{x}dx} = \frac{1}{x}$

So, $\frac{1}{x}.2 = \int \frac{(1 - In\, x)}{x^2}dx + c$

$= -\int \frac{Inx}{x^2}dx + \int \frac{1}{x^2}dx + c$

$= \frac{In\, x}{x} + \frac{1}{x} - \frac{1}{x} + c$

$In\, y = In\, x + cx$

$y(1) = 3$

$\Rightarrow In\, 3 = c$

$In\, y = In\, x + x\, In\, 3$

$y = 3^x.x$

$y(3) = 3^3.3 = 81$

37. Solution: (a)

$y" - 4y' + 4y = 0$

$(D - 2)^2 y = 0$

$y' = (c_1 + c_2 t)e^2 t$

$at\ t = 0, y = 1$

$c_1 = 1$

$y' = 2(c_1 + c_2 t)e^{2t} + c_2 e^{2t}$

$= (2c_1 + c_2 + 2c_2 t)e^{2t}$

$y' = 0 at\ t = 0$

$2c_1 + c_2 = 0 \qquad c_2 = -2c_1 = -2$

$y = (1 - 2t)e^{2t}$

$y\left(-\frac{1}{2}\right) = \frac{2}{e}$

38. Solution: (c)

$P_n(x)$ is basically Hermite polynomial which is even if n is even and odd when n is odd. Making use of this property we can clearly see that (a), (b), (d) are not Hermite polynomial. So (c) is only correct answer.

39. Solution: (d)

$x\frac{dy}{dx} - xy = \exp(x)$

$\frac{dy}{dx} - y = \frac{e^x}{x}$

$IF = e^{-x}$

$y.e^{-x} = \int \frac{e^x}{x}.e^{-x}dx + c$

$= \log x + c$

$y = e^x \log x + e^{x+1}$

At $n = 1, y = e^2 c = e$

So, $\qquad y = e^x \log x + e^{x+1}$

40. Solution: (a)

Given differential is $ds = (1 + y^2)dx + xydy$

$$xds = x(1 + y^2)dx + x^2 ydy$$

$= \frac{1}{2}[2x(1 + y^2)dx + 2x^2 ydy]$

$= \frac{1}{2}d(x^2(1 + y^2))$

$d\left[\frac{1}{2}x^2(1 + y^2)\right\}$