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Mathematical Methods in Physics

CHAPTER 1

Unit 1.1: Dimensional Analysis

1. Using dimensional analysis, find the correct option corresponding to pressure (where h is Planck's constant, v is velocity and x is displacement)
(a) hv / x^4 (b) $h^{0.5}v / x^4$
(c) $h^{0.5}v / x^6$ (d) $h^{0.5}v^2 / x^4$
2. From the given options, choose the pair in which the physical quantities have identical dimensions:
(a) Young's modulus and Planck's constant
(b) Angular momentum and Impulse
(c) Young's modulus and Impulse
(d) Pressure and Young's Modulus
3. If the velocity v, force F and area A are taken as the base units, then which of the following options corresponds to the mass density:
(a) $F / A v^2$ (b) Fv / A^3
(c) F / A^2v (d) Av^2 / F^2
4. Using dimensional analysis, find the correct corresponding to the gravitational constant G (where k_b = Boltzmann constant, T = temperature, h = Planck's constant, m = mass and c = speed of light)
(a) hc^2 / m (b) $hc^5 / k_b T$
(c) k_bT^2 / h^2c^3 (d) hc / m^2
5. Using dimensional analysis, find the correct option corresponding to moment of inertia (where J is impulse, v is velocity and A is area)
(a) $J / A v^2$ (b) $J A / v$
(c) $J A^2 / v$ (d) Av^2 / J^2

ANSWER KEY				
1	2	3	4	5
a	d	a	d	b

:: Solutions ::

1. Since, dimensions of pressure is given by: -
Force =pressure ×Area=mass acceleration.
or $p = \frac{mass \times accelaration}{area} = ML^{-1}T^{-2}$
and h (Planck's constant) has the dimension of angular momentum i.e., $M\,L^2T^{-1}$
and velocity as: LT^{-1}
and displacement as: L
Clearly, from options, $\frac{hv}{x^4}$ will give $\frac{ML^2T^{-1}.LT^{-1}}{L^4}$ or $ML^{-1}T^{-2}$
which is the same as the dimensions of pressure.
∴ option (a) is correct.

2. Since the dimension of Planck's constant is same as that of angular momentum i.e. ML^2T^{-1} and the dimension of Young's Modulus $i.e\ \frac{F}{A}.\frac{\Delta l}{l}\quad is\ \ ML^{-1}T^{-2}$
Impulse $= \int F\,dt = MLT^{-1}$
Pressure $= \frac{F}{A} = ML^{-1}T^{-2}$
∴ option (d) is correct.

3. Since, mass density i.e. $\frac{Mass}{Volume}$ has dimension ML^{-3}
Also, the dimension of: -force is MLT^{-2}
Area (A) is L^2
Velocity (v)is LT^{-1}
So, clearly $\frac{F}{Av^2} = MLT^{-2}\ L^{-2}\ L^{-2}T^2 = ML^{-3}$
∴ option (a) is correct.

4. Since Energy of different forms can be written as $= \frac{1}{2}\,mv^2$
$=$ k$_b$T

$$= \frac{-GMm}{r}$$

$\therefore G = \frac{(Energy).L}{M^2}$ gives $ML^2T^{-2}.L.M^{-2}$

$$= M^{-1}L^3T^{-2} \qquad -(1)$$

Also, we known Planck's constant (h) has dimensions ML^2T^{-1}1
and c has dimension LT^{-1}

$$\therefore \frac{hc}{m^2} = ML^2T^{-1}.LT^{-1}.M^{-2}$$

$$= M^{-1}L^3T^{-2} \qquad -(2)$$

∴ from (1) & (2), option (d) is correct.

5. Since we know that the dimensions of moment of inertia is ML^2 & that of velocity is LT^{-1}
Area is L^2
Impulse is MLT^{-1}
So, clearly. $\frac{Impulse}{Velocity}$ gives M as the resulting dimension.
So, by multiplying $\frac{Impulse}{Velocity}$ with area, we will have the same dimensions as that of moment of inertia i.e. ML^2
∴ option (b) is correct.

Unit 1.2: Vector Algebra

1. A unit vector $\hat{n}$ in x-y plane is at angle 150° w.r.t$\hat{\imath}$. The angle between the vectors $\vec{u} = a\hat{\imath}$ +b$\hat{n}$ and $\vec{v} = b\hat{\imath}$ +a$\hat{n}$ will be 120°. Then a^2+b will approximately be
(a) 3ab (b) 2ab
(c) ab (d) 4ab

2. Let $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$ and $\vec{b} = \hat{\imath} + \hat{\jmath}$. If $\vec{c}$ is such that $\vec{a} * \vec{c} = c$ and $|\vec{c} - \vec{a}| = 2$, then the value of |c| will be
(a) $1 \pm \sqrt{2}$ (b) 0
(c) $\sqrt{2}$ (d) $\sqrt{3}$

3. Let $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$ and $\vec{b} = \hat{\imath} + \hat{\jmath}$. If $\vec{c}$ is such that $\vec{a} \times \vec{c} = c$ and $|\vec{c} - \vec{a}| = 2$ and the angle between ($\vec{a}\ x\ \vec{b}$) and $\vec{c}$ is 30°. Then | ($\vec{a}\ x\ \vec{b}$) x $\vec{c}$ | will be
(a) $\frac{1-\sqrt{3}}{2}$ (b) $\frac{1\pm\sqrt{2}}{\sqrt{2}}$
(c) $\frac{\sqrt{3}}{2}$ (d) 0

4. Given that for three vectors $\vec{a}, \vec{b}$ and $\vec{c}$, $\vec{a}\,.(\vec{b}\ x\ \vec{c})$ = 4, then what will be $\vec{c}\,.(\vec{a}\ x\ \vec{b})$
(a) 2 (b) 4
(c) 8 (d) 12

5. Given that $\vec{a} = \hat{\imath} + \hat{\jmath}$, $\vec{b} = \hat{\jmath} + \hat{k}$ and $\vec{c} = \hat{\imath} + \hat{k}$, then $\vec{a}\ x\ \vec{b}\ x\ \vec{c}$ will be
(a) $-\hat{\imath} + \hat{\jmath}$ (b) $\hat{\imath} + \hat{\jmath}$
(c) $\hat{\imath} + \hat{\jmath} + \hat{k}$ (d) $\hat{\jmath} + \hat{k}$

6. Given that $\vec{a} = \hat{\imath} + \hat{\jmath}$, $\vec{a}.\vec{c} = \vec{a}.\vec{b}$ and $\vec{b} = \vec{c}$, then $\vec{a}\ x\ \vec{b}$ x $\vec{c}$ =
(a) 4 (b) 3
(c) 6 (d) 0

7. Consider three-unit vectors $\vec{a}, \vec{b}$ and $\vec{c}$ such that $\vec{a}\ x\ \vec{b}\ x\ \vec{c} = \frac{\vec{b}-\vec{c}}{2}$, then what will be the angle (in degrees) that $\vec{a}$ makes with $\vec{b}$
(a) 45 (b) 30
(c) 60 (d) 90

8. Consider three-unit vectors $\vec{a}, \vec{b}$ and $\vec{c}$ such that $\vec{a}\ x\ \vec{b}\ x\ \vec{c} = \frac{\vec{b}}{2}$ - m$\vec{c}$, what will be the value of m such that the angle between $\vec{a}$ and $\vec{b}$ is 120°
(a) $\frac{1}{3}$ (b) $\frac{1}{2}$
(c) $\frac{1}{9}$ (d) -2

9. Consider that the equation of a surface is given by $x^2 + 2y^2 + 3z^2 = 0$, then the unit normal at the point (1, 1, 0) is
(a) $\frac{2\hat{\imath} + \hat{\jmath}}{\sqrt{3}}$ (b) 0
(c) $\frac{\hat{\imath} - 2\hat{\jmath}}{\sqrt{3}}$ (d) $\frac{\hat{\imath} + 2\hat{\jmath}}{\sqrt{5}}$

9. Given that the two surfaces $ax^2 + by^2 + z^2 = 0$ and $2ax^2 - by^2 = 0$ at the point (1,1,1) are orthogonal to each other. Find the relation between a and b
(a) a = b (b) a =2b
(c) b =2a (d) b = $\sqrt{2}$ a

11. Given that the equation of surface is $x^2 + y^2 + z^2 = 0$, then equation of tangent plane to the surface at (1, 2, 1) will be
(a) $x + y + z = 4$ (b) $x + 2y + z = 6$
(c) $x + y + z = 0$ (d) $x + y = 6$

12. Given that the equation of surface is $x^2\,y\,z = 2$, then equation of tangent plane to the surface at (1, 1, 1) will be
(a) $x + 2y + z = 4$
(b) $2x + y + z = 4$
(c) $x + 2y + 4z = 6$
(d) $x + y = 0$

13. Find the value of directional derivative of the surface $x^2 + y^2 + z^2 = 0$ in the direction of $\vec{A} = \frac{\sqrt{3}}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath}$ at (1,1,0)
(a) $\sqrt{3} + 1$ (b) $\sqrt{3}$
(c) 0 (d) 1

14. The directional derivative of scalar function $\Phi(x, y) = \frac{x}{x^2 + y^2}$ along a line makes 60° angle with positive x-axis at (1,1) is
(a) $\frac{\sqrt{3}}{2}$ (b) $-\frac{\sqrt{3}}{2}$
(c) 2 (d) -2

15. Find the angle (in degrees) w.r.t. the tangent of the surface at which the direction along which the directional derivative is 0
(a) 45 (b) 60
(c) 30 (d) 90

16. Find the angle (in degrees) w.r.t. the tangent of the surface at which the direction along which the directional derivative is maximum
(a) 0 (b) 60
(c) 45 (d) 75

17. Which among the following tells us the rate of change of the surface/scalar function perpendicular to the surface itself
(a) Divergence (b) Curl
(c) Gradient (d) None of these

18. The divergence of a vector function tells us about
(a) Net outward flux (b) Rotational Effect
(c) both a and b (d) none of these

19. For a vector field $\vec{A} = \hat{r}\,r^2$, then divergence of $\vec{A}$ will be
(a) 9 r^2 (b) 3 r^2
(c) 5 r^2 (d) 2 r^2

20. for a constant vector field $\vec{a}$, the value of $\nabla \times \vec{a} \times \vec{r}$ will be
(a) $4\vec{a}$ (b) 0
(c) $\vec{a}$ (d) $2\vec{a}$

21. for a constant vector field $\vec{a}$, the value of $\nabla (\vec{a} \cdot \vec{r})$ will be
(a) $4\vec{a}$ (b) 0
(c) $\vec{a}$ (d) $2\vec{a}$

22. Consider a conservative field $\vec{A}$. What will be the work done by the conservative field along a closed curve $x^2 + y^2 = 1$
(a) 0 (b) -2
(c) 2 (d) ½

23. For a vector field $\vec{A} = \frac{-y\,\hat{\imath} + x\,\hat{\jmath}}{x^2 + y^2}$, what will be the work done by $\vec{A}$ over a square having sides 1 unit and centre at (1,1)
(a) 13 (b) 9
(c) 0 (d) -25

24. For a vector field $\vec{F} = \frac{-y\,\hat{\imath} + x\,\hat{\jmath}}{x^2 + y^2}$, what will be the work done by $\vec{A}$ over a curve $x^2 + y^2 = 12$
(a) 0 (b) π
(c) 2 π (d) - π

25. The vector field $\vec{A} = -(x^2\,\hat{\imath} + y\,\hat{\jmath})$ is:
(a) Solenoidal
(b) Irrotational
(c) both a and b
(d) none of these

26. $\vec{A} = (-mx\,\hat{\imath} + y\,\hat{\jmath})$, for what value of m, will be the $\vec{A}$ solenoidal
(a) 0 (b) 1
(c) -2 (d) 4

27. What should be the value of m, such that the vector field $\vec{A} = (-mx\,\hat{\imath} + y\,\hat{\jmath})$ in the region $(x-2)^2 + (y-2)^2 = 1$, is irrotational
(a) 2 (b) 1
(c) 0 (d) -1

28. For the constant vector $\vec{a}$, the force field is defined as $\vec{F} = \vec{r} \times \vec{a}$, find the work done by the force field over the closed loop $x^2 + y^2 = 1$
(a) 2 πa (b) -2πa
(c) 2 π (d) -0

29. Find the work done by the vector field
$\vec{A} = ((xy + y^2)\,\hat{\imath} + x^2\,\hat{\jmath})$
Over the curve between y=x and $y^2 = 4ax$
(a) $7\,a^2$ (b) $6.4\,a^3$
(c) $2\,\pi a^2$ (d) 0

30. Given that $\vec{F} = 4x\,\hat{\imath} - 2\,y^2\,\hat{\jmath} + z^2\,\hat{k}$, surface of cylinder: $x^2 + y^2 = 4$, $z = 0$ to $z = 3$. Then find the flux through this surface
(a) 0 (b) 20π
(c) 84 π (d) 60 π

ANSWER KEY

1	2	3	4	5	6	7	8	9	10
a	a	b	b	a	d	c	b	d	d
11	12	13	14	15	16	17	18	19	20
b	c	a	b	d	a	a	a	c	d
21	22	23	24	25	26	27	28	29	30
c	a	c	c	b	b	d	b	b	c

:: Solutions ::

1. $\vec{u}.\vec{v} = uv\ cos\ cos\ 120^\circ = uv\left(\frac{-1}{2}\right)$

   Or $(a\hat{\imath} + b\hat{n}).(a\hat{n} + b\hat{\imath}) = \frac{1}{2}uv$

   Or $a^2\ cos\ cos\ 150^\circ + ab + ab + b^2\ cos\ cos\ 150^\circ =$

   $\sqrt{a^2 + b^2 + 2ab\ cos\ cos\ 150^\circ}$

   $\sqrt{a^2 + b^2 + 2ab\ cos\ cos\ 150^\circ}$

   Or $(a^2 + b^2)\sqrt{3} - 4ab = a^2 + b^2 = \sqrt{3}ab$

   Or $(a^2 + b^2) = ab.\left(\frac{4-\sqrt{3}}{\sqrt{3}-1}\right) \approx 3ab$

2. Given $\vec{a}.\vec{c} = 0$ or $ac\ cos\ cos\ \emptyset = c$

   Or $a\ cos\ cos\ \emptyset = 1$ ---- (1)

   & $|\vec{c} - \vec{a}| = 2$ or $\sqrt{a^2 + c^2 - 2ac\ cos\ cos\ \theta} = 2$

   Using (1), $\sqrt{a^2 + c^2 - 2c} = 2$

   Or $a^2 + c^2 - 2c = 4$

   Where $|a|^2 = 3$

   Or $c = 1 + \sqrt{2}$

3. $\left|\vec{a} \times \vec{b} \times \vec{c}\right| = \left|\vec{a} \times \vec{b}\right| |\vec{c}|\ sin\ 30^\circ$

   $= |-i + j||\vec{c}|\ sin\ sin\ 30^\circ$

   $=(\sqrt{2})(1 \pm \sqrt{2})\frac{1}{2}$

   $= \frac{1\pm\sqrt{2}}{\sqrt{2}}$

4. Since we know that the scalar triple product of 3 vector $\vec{a}, \vec{b}, \vec{c}$ are

   $\vec{a}.(\vec{b} \times \vec{c}) = \vec{b}(\vec{c} \times \vec{a}) = \vec{c}(\vec{a} \times \vec{b})$ and

   given: $\vec{a}.(\vec{b} \times \vec{c}) = 4$

   $\therefore \vec{C}.(\vec{a} \times \vec{b}) = 4$

5. Since we know that: $a \times b \times c = \vec{b}(\vec{a}.\vec{c}) - \vec{c}(\vec{a}.\vec{b})$

   $= (\hat{\jmath} + \hat{k})(1) - (\hat{\imath} + \hat{k})(1)$

   $= \hat{\jmath} = \hat{\imath}$

6. Since $\vec{a} \times \vec{b} \times \vec{c} = \vec{b}.(\vec{a}.\vec{c}) - \vec{c}(\vec{a}.\vec{b})$

   & given that $\vec{a}.\vec{c} = \vec{a}.\vec{b}$ & $\vec{b} = \vec{c}$

   $\vec{a} \times \vec{b} \times \vec{c} = \vec{b}.(\vec{a}.\vec{c}) - (\vec{b} - \vec{c})$

   $= 0$

7. Since $\vec{a} \times \vec{b} \times \vec{c} = \vec{b}.(\vec{a}.\vec{c}) - \vec{c}(\vec{a}.\vec{b})$

   Given: $\vec{a} \times \vec{b} \times \vec{c} = \frac{\vec{b}}{2} - \frac{\vec{c}}{2}$

   Or $\vec{a}.\vec{b} = \frac{1}{2}$

   $ab\ cos\ cos\ \theta = \frac{1}{2}$

   & given that $|a| = |b| = 1 \therefore \theta = 60^\circ$

8. Given $\vec{a} \times \vec{b} \times \vec{c} = \frac{\vec{b}}{2} + m\vec{c}$

   Since $\vec{a} \times \vec{b} \times \vec{c} = \vec{b}(\vec{a}.\vec{c}) - \vec{c}(\vec{a}.\vec{b})$

   On comparing, we have: $-(\vec{a}.\vec{b}) = m$

   Or $a\ b\ cos\ cos\ \theta = -m$

   Or $1.1.cos\ 120' = -m$

   Therefore m = 1/2

9. Given $x^2 + 2y^2 + 3z^2 = 0 \quad \therefore \emptyset = (x^2 + 2y^2 + 3z^2)$

   The unit normal is $\hat{n} = \frac{\vec{\nabla}\emptyset}{|\vec{\nabla}\emptyset|}$

   Where $\emptyset = x^2 + 2y^2 + 3z^2$

   $= 2x\hat{\imath} + 4y\hat{\jmath} + 6z\hat{k}$

   $\therefore \left|\vec{\nabla}\emptyset\right| = \sqrt{4x^2 + 16y^2 + 36z^2}$

   $\hat{n} = \frac{\vec{\nabla}\emptyset}{|\vec{\nabla}\emptyset|}_{(1,1,0)} = \frac{2\hat{\imath}+4\hat{\jmath}}{\sqrt{20}} = \frac{\hat{\imath}+2\hat{\jmath}}{\sqrt{5}}$

10. Given that the two surfaces are orthogonal to each other. So their normal must also be orthogonal i.e. Perpendicular

    $\therefore \vec{\nabla}\emptyset = \vec{\nabla}(ax^2 + by^2 + z^2) = 2ax\hat{\imath} + 2by\hat{\jmath} + 2z\hat{k}$

    $\vec{\nabla}\emptyset_2 = \vec{\nabla}(2ax^2 - by^2) = 40x\hat{\imath} - 2by\ \hat{\jmath} + 0\hat{k}$

    $\vec{\nabla}\emptyset_1.\vec{\nabla}\emptyset_2|_{(1,1,1)} = 0$ or $8ax^2 - 4by^2|_{(1,1,1)} = 0$

    $\therefore b = \sqrt{2}a$

11. Here $\overrightarrow{P\theta}$ is tangent to the surface and $\vec{\nabla}\emptyset$ is normal to the surface. $\therefore \vec{\nabla}\emptyset,.(\overrightarrow{P\theta}) = 0$

    Where, $\vec{\nabla}\emptyset|_P = \vec{\nabla}(x^2 + y^2 + z^2)|_{1,2,3}$

    $= 2x\hat{\imath} + 2y\hat{\jmath} + 2 = R$

    $= 2\hat{\imath} + 2y\hat{\jmath} + 2 = R$

    & $\overline{PO} = (x-1)\hat{\imath} + (y-2)\hat{\jmath} + (z-1)\hat{k}$

    $\therefore \vec{\nabla}\emptyset_p.\overline{P\theta} = 0(2\hat{\imath} + 4\hat{\jmath} + 2R).\big((x+1)\hat{\imath} + (y-2)\hat{\jmath} +$

    $(z-1)i\big) = 0$

    $or\ \ x + 2y + z = 0$

12. $\vec{\nabla}\emptyset_p.\overline{P\theta} = 0$

    Where $\overline{P\theta} = (x-1)\hat{\imath} + (y-1)j + (z-1)\hat{k}$

    $\therefore\ \vec{\nabla}\emptyset|_p\overline{P\theta} = 0$ or $2x - 2 + y - 1 + z - 1 = 0$

    $or\ \ 2x + y + z = 4$

13. Directional derivative of $\vec{A} = \vec{\nabla}\emptyset|.\vec{A}$

    $= (2x\hat{1} + 2y\hat{\jmath} + 2z\hat{k})|_{(1,1,0)}.\left(\frac{\sqrt{3}}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath}\right)$

    $= \sqrt{3} + 1 + 0$

    $= \sqrt{3} + 1$

14. $\hat{A} = cos\ cos\ 60^\circ\hat{\imath} + sin\ sin\ 60^\circ\ \hat{\jmath}$

    $= \frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath}$

    $\therefore D.D.A = \vec{\nabla}\emptyset|_{(1,1)}\hat{A} = \left|\frac{x^2+y^2-x(2x)\hat{\imath}-x.2y\hat{\jmath}}{(x^2+y^2)}\right|_{(1,1)}(\frac{1}{2}\hat{\imath} +$

    $\frac{\sqrt{3}}{2}\hat{\jmath})$

    $=\left(\frac{(1+1+2)\hat{\imath}-2\hat{\jmath}}{2}\right).(\frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath})$

    $= -\frac{\sqrt{3}}{2}$

15. Since, Directional derivative along $\hat{A} = \vec{\nabla}\emptyset|_P\hat{A}$

    $= \left|\vec{\nabla}\emptyset\right|\left|\hat{A}\right|\ cos\ cos\ \theta$

    $=\left|\vec{\nabla}\emptyset\right|\ cos\ cos\ \theta$

    $\therefore D.D_A = 0$ at $\theta = 90^\circ$ from the tangent of the surface.

16. Directional derivative of along $\hat{A} = \vec{\nabla}\emptyset . \hat{A}$
$= |\vec{\nabla}\emptyset||\hat{A}| \cos\cos\theta$
$=|\vec{\nabla}\emptyset| \cos\cos\theta$
$\therefore$ D.D will be maximum for $0 = 0^{\circ}$
17. Since Divergence of $\emptyset = \vec{\nabla}\emptyset$ or $\frac{\partial\emptyset}{\partial x}\hat{\imath} + \frac{\partial\emptyset}{\partial y}\hat{\jmath} + \frac{\partial\emptyset}{\partial z}\hat{k}$
So clearly divergence for a surface /scalar function tells us the rate of change of the scalar function normal to the surface
18. Since, Divergence of $\emptyset = \vec{\nabla}\emptyset = \frac{\partial\emptyset}{\partial x}\hat{\imath} + \frac{\partial\emptyset}{\partial y}j + \frac{\partial\emptyset}{\partial z}\hat{k}$
If $\vec{\nabla}\emptyset = 0 \rightarrow$ not net outward flux $\rightarrow$ solenoidal field
If $\vec{\nabla}\emptyset =$ positive $\rightarrow$ source field
If $\vec{\nabla}\emptyset$ = negative $\rightarrow$ sink field
19. $\vec{\nabla}.\vec{r}.\vec{r} = r^2.\vec{\nabla}.\vec{r} + \vec{\nabla}\vec{r}.\vec{r}$
$= 3r^2 + 2r\hat{r}.\vec{r}$
$= 3r^2 + 2r^2 = 5r^2$
20. Using: $\vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{B}.\vec{\nabla})\vec{A} - (\vec{A}.\vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla}.\vec{B}) - \vec{B}(\vec{\nabla}.\vec{A})$
$\therefore$ For constant vector field $\vec{a}$
$\vec{\nabla} \times \vec{a} \times \hat{r} = (\vec{r}.\vec{\nabla})\vec{a} - (\vec{a}.\vec{\nabla})\vec{r} + \vec{a}(\vec{\nabla}.\vec{r}) - \vec{r}(\vec{\nabla}.\vec{a})$
$= 0 - \vec{a} + 3\vec{a} - 0$
$= 2\vec{a}$
21. $\vec{\nabla}(\vec{a}.\vec{r}) = \vec{\nabla}(a_x x + a_y y + a_z z)$
$\frac{\partial}{\partial x}(a_x x)\hat{\imath} + \frac{\partial}{\partial y}(a_y y)\hat{\jmath} + \frac{\partial}{\partial z}(a_z z)\hat{k}$
$= a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$
$= \vec{a}$
22. Since we know that the work done by a continuous force field along a closed path is zero, whatever be the form of the closed curve.
23. Since we know that the work done by continuous force field along a closed path is zero, whatever be the form of the closed curve. But it is valid only when, within the closed curve, the vector field is defined at all points
Here $\vec{A}$ is not defined only at origin which is not within the curve region
24. The given vector field is conservation but since within the curve, it is not defined at the origin S, we have the work done won't be zero
So, we have to calculate:
$$\oint_0 \vec{F}.\overrightarrow{\mathrm{dr}} = \oint_c \frac{-ydx + xdy}{x^2 + y^2}$$
$$= \int_0^{2\pi} \frac{-R\sin\theta\,(-R\sin\theta\,)d\theta + R\cos\theta\, R\cos\, d\theta}{R^2}$$
$$= \int_0^{2\pi} \mathrm{d}\,\theta \;\; = 2\pi$$
25. Since, $\vec{\nabla}.\vec{A} = -2x + 1$ *which is non zero*
$$and\ \vec{\nabla}\ x\ \vec{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x^2 & -y & 0 \end{vmatrix} = 0$$
$\therefore \hat{A} = -(x^2\hat{\imath} + y\hat{\jmath})$ is irrotational.
26. Condition of $\vec{A}$ being solenoidal field is $\vec{\nabla}.\vec{A} = 0$ or $-m + 1 = 0$
$\therefore m = 1$ is the required value of m
27. For a vector field to be irrational
$$\vec{\nabla} \times \vec{A} = 0 \text{ or } \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -my & x & 0 \end{vmatrix} = 0\ or\ \hat{\imath}(0) - \hat{\jmath}(0) + \hat{k}(1 + m) = 0$$
Or $m = -1$
28. $\oint_c .\vec{F}.\overrightarrow{dr} = \iint_s (\vec{\nabla} \times \vec{F})\ .\ d\vec{s}$ Using stroke's theorem
Where $\vec{\nabla} \times \vec{F} = \vec{\nabla} \times \vec{r} \times \vec{a}$
$= -\vec{\nabla} \times \vec{a} \times \vec{r}$ ($\therefore \vec{\nabla} \times \vec{a} \times \vec{r} = 2\vec{a}$ Where $\vec{a} =$ constant vector)
$\int_S (\vec{\nabla} \times \vec{F}).d\vec{s} = -2a_n \int\int ds$
$= -2a_k\pi(1)^2$
$= -2a_k\pi$
29. For $C1: x = y \therefore dx = dy$
$\therefore \oint_{c_1} \vec{F}.d\vec{r} = \int_{x=0}^{4a}(x^2 + x^2)dx + x^2dx$
$= -64a^3$
For C2: $x = \frac{y^2}{4a} \therefore dx = \frac{y}{2a}dy$
$\int_{C2} = \vec{F}.d\vec{r} = \int_{y=0}^{4x}(y.\frac{y^2}{4a} + y^2)\frac{y}{2a}dy + \frac{y^n}{16a^2}dy$
$= \int_{y=0}^{4a}\left[\frac{y^4}{8a^2} + \frac{y^4}{16a^2} + \frac{y^3}{2a}\right]dy$
$= \frac{3}{16a^2}\left(\frac{y^5}{5}\right) + \frac{1}{2a}\frac{y^4}{4}|_0^{4a} = 38.4a^3 + 32a^3$
= 70.4 a^3
$\oint \vec{F}.d\vec{r} = \int_{c_1}\vec{F}.d\vec{r} + \int_{c_2}\vec{F}.d\vec{r} = 70.4a^3 + (-64a^3)$
= 6.4 a^3
30. Flux = $\oiint_s \vec{F}.d\vec{s} = \iiint_v (\vec{\nabla}.\vec{F})dV$
$= \iiint_{0,0,0}^{2,2\pi,3}(4 - 4y + 2z)dV$
$= \iiint_{0,0,0}^{2,2\pi,3}(4 - 4y + 2z)sds\,d\emptyset\,dz$
$= \iiint_{0,0,0}^{2,2\pi,3}(4 - 4s\sin\sin\emptyset\, + 2z)sds\,d\emptyset\,dz$
Where the limit will be -: $s: 0 \rightarrow 2, \emptyset: 0 \rightarrow 2\pi, z: 0 \rightarrow 3$
$= \int_{s=0}^{2}\int_{\emptyset=0}^{2\pi}\int_{z=0}^{3}(4 - 4\sin\sin\emptyset\, + 2z)\,sds\,d\emptyset\,dz$
= 84 π

Unit 1.3: Matrices

1.3.1: Matrices

1. What are the only possible eigenvalues of a 2x2 idempotent matrix?
(a) 0 and 3 (b) 1 and 2
(c) 0 and 1 (d) none of these

2. For a n x n nilpotent matrix M, one of the eigenvalues have to be
(a) 1 (b) 0
(c) n (d) none of these

3. Which of the following set of vectors are linearly dependent
(a) $\begin{bmatrix}0\\-5i\\-i\end{bmatrix},\begin{bmatrix}5i\\0\\-2i\end{bmatrix}$ and $\begin{bmatrix}i\\2i\\0\end{bmatrix}$
(b) $\begin{bmatrix}0\\5i\\-i\end{bmatrix},\begin{bmatrix}i\\1\\-2i\end{bmatrix}$ and $\begin{bmatrix}5i\\2i\\i\end{bmatrix}$
(c) $\begin{bmatrix}0\\5i\\-i\end{bmatrix},\begin{bmatrix}5i\\0\\-2i\end{bmatrix}$ and $\begin{bmatrix}i\\i\\0\end{bmatrix}$
(d) $\begin{bmatrix}2i\\-5i\\-i\end{bmatrix},\begin{bmatrix}i\\0\\-2i\end{bmatrix}$ and $\begin{bmatrix}i\\2i\\0\end{bmatrix}$

4. Consider A and B be the symmetric matrix such that commutator of A and B is 0, then $[AB]^T$ will be
(a) AB (b) BA
(c) $[AB]^{-1}$ (d) $[BB]^{-1}$

5. For a 3x3 symmetric matrix M, the number of independent components will be
(a) 4 (b) 3
(c) 6 (d) 9

6. For a 3x3 skew symmetric matrix M, the number of independent components will be
(a) 4 (b) 3
(c) 6 (d) 9

7. For $k = a + ib$, where a and b are non-zero constants. What will be the nature of (kM), considering M being a Hermitian matrix
(a) Hermitian
(b) Skew Hermitian
(c) None of these
(d) Det =0

8. Given that a matrix M has its determinant 3, then what will be the value of determinant of inverse of M
(a) $\frac{1}{3}$ (b) 3
(c) $\frac{1}{9}$ (d) 9

9. Consider a Matrix M, such that all of its elements are given by a_{ij}= -ij, Given that M=$\begin{pmatrix}-1 & -2\\-2 & -4\end{pmatrix}$
(a) $0, -1$ (b) $-1, -5$
(c) $5, 0$ (d) 0, -5

10. For a matrix, the necessary condition is that each row and each column is normalized, the nature of matrix will be
(a) Orthogonal
(b) Skew Hermitian
(c) Hermitian
(d) Idempotent

11. The eigenvalues of the matrix M=$\begin{pmatrix}cos\theta & -sin\theta\\sin\theta & cos\theta\end{pmatrix}$ at θ=30°
(a) $\frac{\sqrt{3}}{2}\pm\frac{i}{2}$ (b) $\frac{i}{2}\pm\frac{\sqrt{3}}{2}$
(c) $\sqrt{2}+i$ (d) $i\pm\sqrt{2}$

12. For a Matrix M,($i(M-M^T))^T$ is
(a) Orthogonal (b) Hermitian
(c) Identical (d) Idempotent

13. What will be the commutator of two matrices B=$\begin{bmatrix}0 & 0 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$ and C=$\begin{bmatrix}0 & 0 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$
(a) BC (b) CB
(c) B^{-1} (d) 0

14. What will be the commutator of two matrices
B=$\begin{bmatrix}0 & -1 & 0\\1 & 0 & 0\\0 & 0 & 0\end{bmatrix}$ and C=$\begin{bmatrix}0 & 0 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$
(a) $\begin{bmatrix}0 & 0 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$ (b) $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$
(c) $\begin{bmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$ (d) $\begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$

15. Consider A to be a 3x3 Hermitian matrix having trace=5, then number of independent parameters of A is
(a) 7 (b) 5
(c) 8 (d) 9

16. The characteristic equation of a 3x3 matrix is A^3-$3A^2$+3A+7I=0. Then Det (A^T) will be
(a) -2 (b) -7
(c) 2 (d) 7